Complex Analysis

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July 14, 2023

Abstract

The main reference for these notes is [Con78], which I find much more readable than [Ahl66].

Contents

Chapter 1

Introduction

1.1 Preliminaries

Definition 1.1. Let $\{a_n\}$ be a real sequence. Define the limit superior and the limit inferior of a sequence to be

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \dots\}$$
$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\}$$

Proposition 1.2. $\mathbb{C}$ *is complete.*

Proof. Let $\{z_n = x_n + \imath y_n\}$ be a Cauchy sequence in $\mathbb{C}$. For every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|z_n - z_m| < \varepsilon$, and thus, $|x_n - x_m| < \varepsilon$ and $|y_n - y_m| < \varepsilon$. Consequently, both the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy and converge and therefore, so does $\{z_n\}$. ■

1.2 Power Series

Definition 1.3 (Power Series). Let $a \in \mathbb{C}$. A power series about a is an infinite series of the form $\sum_{n=0}^{\infty} a_n(z-a)^n$ where $\{a_n\}$ is an infinite sequence of complex numbers.

Example 1.4. The power series $\sum_{n=0}^{\infty} z^n$ converges if $|z| < 1$ and diverges if $|z| > 1$.

Proof. Suppose $|z| < 1$. We shall show that the sequence of partial sums is Cauchy. Indeed, for $m \geq n$, we have

$$|z^n + \cdots + z^m| < |z|^n \frac{1}{1-|z|}$$

On the other hand, if $|z| > 1$, we shall show that the sequence is not Cauchy. If s_n denotes the n-th partial sum of the series, we note that

$$|s_{n+1} - s_n| = |z|^{n+1}$$

This completes the proof. ■

Theorem 1.5. *For a given power series* $\sum_{n=0}^{\infty} a_n(z-a)^n$*, define the number* $R \in [0, \infty]$ *by*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$$

then

(a) if $|z-a| < R$*, the series converges absolutely*

(b) if $|z-a| > R$*, the series diverges*

(c) if $0 < r < R$*, then the series converges uniformly on* $\overline{B}(a,r)$

This R is known as the radius of convergence of the power series.

Proof. For simplicity, let $a = 0$ (this does not affect the correctness of the proof).

(a) Since $|z| < R$, there is a real number r such that $|z| < r < R$. Consequently, by definition, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n|^{1/n} < \frac{1}{r}$. In other words, for all $n \geq N$, $|z|^n|a_n| < 1$. It is evident from here that the partial sums form a Cauchy sequence.

(b) If $|z| > R$, there is a positive real number r such that $|z| > r > R$, consequently, there is a subsequence $\{n_k\}$ such that $|a_{n_k}|^{1/n_k} r > 1$. If A_n denotes the partial sums of the sequence, then $|A_{n_k} - A_{n_k - 1}| > 1$ and thus, the sequence is not Cauchy, and therefore, divergent.

(c) There is a positive real number ρ such that $r < \rho < R$ and a natural number N such that for all $n \geq N$, $|a_n| < \frac{1}{\rho^n}$. Consequently, for all $z \in \overline{B}(0,r)$, $|a_n z^n| < \left(\frac{r}{\rho}\right)^n$ and we are done due to the Weierstrass M-test.

■

Theorem 1.6 (Mertens). *Let* $\{a_n\}$*,* $\{b_n\}$ *and* $\{c_n\}$ *be complex sequences such that*

(a) $\sum a_n$ *converges absolutely and* $\sum b_n$ *converges*

(b) $\sum a_n = A$ *and* $\sum b_n = B$

(c) $\{c_n\}$ *is the Cauchy product of* $\{a_n\}$ *and* $\{b_n\}$

Then, $\sum c_n$ *converges to* AB*.*

Proof. Define A_n, B_n and C_n in the obvious way. Further, let $\beta_n = B_n - B$. Then, we have

$$\begin{aligned} C_n &= \sum_{k=0}^{n} a_k B_{n-k} \\ &= \sum_{k=0}^{n} a_k (B + \beta_{n-k}) \\ &= BA_n + \sum_{k=0}^{n} a_k \beta_{n-k} \end{aligned}$$

Let $\gamma_n = \sum_{k=0}^{n} a_k \beta_{n-k}$. We shall show $\lim_{n\to\infty} \gamma_n = 0$. Let $\varepsilon > 0$ be given. Let $\alpha = \sum_{n=0}^{\infty} |a_n|$ (since it is known that it converges absolutely). From (b), we know that $\beta_n \to 0$, therefore, there is N such that $|\beta_n| < \varepsilon/\alpha$ for all $n \geq N$. Consequently, we have

$$\begin{aligned}|\gamma_n| &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon\alpha\end{aligned}$$

Which immediately gives us

$$\limsup_{n\to\infty} |\gamma_n| \leq \varepsilon\alpha$$

and since ε was arbitrary, we have the desired conclusion. ∎

1.3 Analytic Functions

Definition 1.7. If $G \subset \mathbb{C}$ is open, and $f : G \to \mathbb{C}$ then f is *differentiable* at a point $a \in G$ if

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The value of this limit is denoted by $f'(a)$ and is called the *derivative* of f at a. If f is differentiable at each point of G we say that f is differentiable on G. If f' is continuous then we say that f is *continuously differentiable*.

Proposition 1.8. *If $f : G \to \mathbb{C}$ is differentiable at $a \in G$, then f is continuous at a.*

Proof. One line:

$$\lim_{z\to a} |f(z) - f(a)| = \lim_{z\to a} \frac{|f(z) - f(a)|}{|z-a|} |z-a| = \lim_{z\to a} \left| \frac{f(z) - f(a)}{z-a} \right| \lim_{z\to a} |z-a| = 0$$

∎

Definition 1.9 (Analytic Function). A function $f : G \to \mathbb{C}$ is *analytic* if f is continuously differentiable on G.

Theorem 1.10 (Chain Rule). *Let f and g be analytic on G and Ω respectively and suppose $f(G) \subseteq \Omega$. Then $g \circ f$ is analytic on G and*

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

for all $z \in G$.

Proof. Define the function $h \equiv g \circ f : G \to \mathbb{C}$. We shall show that the function h is differentiable at every point $a \in G$ and that the derivative at a equals $g'(f(a))f'(a)$. Notice that the latter implies analyticity.

Let $z = f(a)$. Then, by definition, we have functions $u : G \to \mathbb{C}$ and $v : \Omega \to \mathbb{C}$ with $\lim_{x\to a} u(x) = 0$ and $\lim_{x\to z} v(z) = 0$ satisfying

$$\begin{aligned}f(x) - f(a) &= (x-a)(f'(a) + u(x)) \\ g(x) - g(z) &= (x-z)(g'(z) + v(x))\end{aligned}$$

Note that

$$\begin{aligned} h(x) - h(a) &= g(f(x)) - g(f(a)) \\ &= (f(x) - f(a))(g'(z) + v(f(x))) \\ &= (x - a)(f'(a) + u(x))(g'(z) + v(f(x))) \end{aligned}$$

Taking the limit gives the desired result. ■

Theorem 1.11. *Let* $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ *have radius of convergence* $R > 0$. *Then*

(a) For each $k \geq 1$, *the series*

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k} \qquad (\star)$$

has radius of convergence R

(b) The function f is infinitely differentiable on $B(a, R)$ *and furthermore,* $f^{(k)}(z)$ *is given by the series* ($\star$) *for all* $k \geq 1$ *and* $|z - a| < R$

(c) For $n \geq 0$,

$$a_n = \frac{1}{n!} f^{(n)}(a)$$

Proof. It suffices to prove the theorem for $a = 0$.

(a) We shall prove it for $k = 1$ since the general case would follow inductively. Since $\lim_{n \to \infty} n^{1/(n-1)} = 1$, it suffices to show that

$$\limsup_{n \to \infty} |a_n|^{1/n} = \limsup_{n \to \infty} |a_n|^{1/(n-1)}$$

Note that we may write

$$f(z) = a_0 + z \underbrace{\sum_{n=1}^{\infty} a_n z^{n-1}}_{g(z)}$$

It is not hard to argue that both $f(z)$ and $g(z)$ have the same radius of convergence, and thus $\limsup |a_n|^{1/n} = \limsup |a_n|^{1/(n-1)}$.

(b) Again, we shall only show this for $k = 1$ since the general case would follow inductively. Define

$$s_n = \sum_{k=0}^{n} a_k z^k \quad \text{and} \quad e_n = \sum_{k=n+1}^{\infty} a_k z^k$$

Obviously, $f = s_n + e_n$ for all $n \in \mathbb{N}$. Let $g(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Let $w \in B(0, R)$ and choose a positive real number r such that $0 < |w| < r < R$. Let $\delta > 0$ be chosen such that $B(w, \delta) \subseteq B(0, r)$. Choose any $\varepsilon > 0$.

Then, we have

$$\frac{f(z) - f(w)}{z - w} - g(w) = \left(\frac{s_n(z) - s_n(w)}{z - w} - g(w) \right) + \frac{e_n(z) - e_n(w)}{z - w}$$

Note that

$$\left|\frac{e_n(z)-e_n(w)}{z-w}\right| \leq \sum_{k=n+1}^{\infty} |z^{k-1}+\cdots+w^{k-1}| \leq \sum_{k=n+1}^{\infty} kr^{k-1}$$

Since the series on the right is the trailing sum of a convergent series, there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $\sum_{k=n+1}^{\infty} kr^{k-1} < \varepsilon/3$.

Similarly, there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|s_n'(w) - g(w)| < \varepsilon/3$. Finally, there is $\delta' > 0$ such that for all $z \in B(w, \delta')$,

$$\left|\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right| < \frac{\varepsilon}{3}$$

Putting these together, we see that for all $z \in B(w, \min\{\delta, \delta'\})$, and $n \geq \max\{N_1, N_2\}$

$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| \leq \left|\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right| + |s_n'(w) - g(w)| + \left|\frac{e_n(z)-e_n(w)}{z-w}\right| \leq \varepsilon$$

And we are done.

(c) Straightforward.

■

Corollary 1.12. If the series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R > 0$ then $f(z)$ is analytic in $B(a, R)$.

1.4 Cauchy Riemann Equations

Let $f : G \to \mathbb{C}$ be analytic and let $u(x,y) = \Re f(x+iy)$ and $v(x,y) = \Im f(x+iy)$. Then, we must have, for all $z \in G$,

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{f(z+ih)-f(z)}{ih}$$

The analyticity of f implies the differentiability of u and v and thus, the above equality is equivalent to

$$u_x + iv_x = f'(z) = \frac{1}{i}\left(u_y + iv_y\right)$$

or,

$$u_x = v_y \qquad \text{and} \qquad u_y + v_x = 0 \tag{\textbf{CR}}$$

Suppose u and v have continuous partial derivatives, in which case, recall that second order mixed derivatives exist and do not depend on the order of derivatives taken, that is, $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

Straightforward algebraic manipulation would yield

$$u_{xx} + u_{yy} = 0$$

In other words, u and v are harmonic conjugates.

Theorem 1.13. *Let $G \subseteq \mathbb{C}$ and $u, v : G \to \mathbb{R}$ have continuous partial derivatives. Then $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic if and only if u and v satisfy (**CR**).*

Proof. Suppose the functions u and v satisfy the hypothesis of the theorem. Let $z = x + iy$. We shall show that

$$\lim_{s+it\to 0} \frac{f(z + (s + it)) - f(z)}{s + it}$$

exists.

Define

$$\varphi(s,t) = \big(u(x+s, y+t) - u(x,y)\big) - \Big(u_x(x,y)s + u_y(x,y)t\Big)$$
$$\psi(s,t) = \big(v(x+s, y+t) - v(x,y)\big) - \Big(v_x(x,y)s + v_y(x,y)t\Big)$$

It is not hard to see, using **CR**, that

$$\varphi(s,t) + i\psi(s,t) = f(z + (s+it)) - f(z) - (s+it)(u_x(x,y) + iv_x(x,y))$$

and hence, it would suffice to show that

$$\lim_{s+it\to 0} \frac{\varphi(s,t) + i\psi(s,t)}{s+it} = 0$$

We have

$$u(x+s, y+t) - u(x,y) = u(x+s, y+t) - u(x, y+t) + u(x, y+t) - u(x,y)$$

Due to the Mean Value Theorem, there are real numbers s_1 and t_1 with $|s_1| < s$ and $|t_1| < t$ such that

$$u(x+s, y+t) - u(x,y) = u_x(x+s_1, y+t)s + u_y(x, y+t_1)t$$

Thus,

$$\varphi(s,t) = (u_x(x+s_1, y+t) - u_x(x,y))s + (u_y(x, y+t_1) - u_y(x,y))t$$

Using continuity, it is not hard to see that

$$\lim_{s+it\to 0} \frac{\varphi(s,t)}{s+it} = 0$$

and a similar result can be obtained for $\psi(s,t)$.

This completes the proof. ■

Theorem 1.14. *Let G be either the whole complex plane $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function then u has a harmonic conjugate.*

Proof. ■

1.5 Analytic Functions as Mappings

We shall suppose in this section that all paths are continuously differentiable.

Theorem 1.15. *If $f : G \to \mathbb{C}$ is analytic, then f preserves angles at each point $z_0 \in G$ where $f'(z_0) \neq 0$.*

Proof. Straightforward. ■

Maps which preserve angles are known as **conformal maps**. Thus, if f is analytic on $G \subseteq \mathbb{C}$ and $f'(z) \neq 0$ for all $z \in G$, it is conformal.

Definition 1.16. A mapping of the form $S(z) = \dfrac{az+b}{cz+d}$ where $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called a *linear fractional transformation*. If a, b, c, d are such that $ad - bc \neq 0$, then $S(z)$ is called a Möbius Transformation.

A Möbius Transformtion is invertible, where

$$S^{-1}(z) = \frac{dz - b}{-cz + a}$$

Chapter 2

Complex Integration

2.1 Riemann Stieltjes Integral

The following definition is taken from [Rud53]

Definition 2.1. Let $[a, b]$ be a given interval. By a partition P of $[a, b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$ where

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b$$

Let $\alpha : [a, b] \to \mathbb{R}$ be monotonically increasing. Corresponding to each partition P of $[a, b]$, write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \quad \text{for } 1 \leq i \leq n$$

Let $f : [a, b] \to \mathbb{R}$ be bounded. For each partition $[x_{i-1}, x_i]$, let

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \qquad m_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$$

Define

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

and

$$\overline{\int_a^b} f \, d\alpha = \inf_{\mathcal{P}} U(P, f, \alpha) \qquad \underline{\int_a^b} f \, d\alpha = \sup_{P \in \mathcal{P}} L(P, f, \alpha)$$

If the above two values are equal, we say that f is *Riemann-Stieltjes integrable* with respect to α on $[a, b]$ and denote the common value as $\int_a^b f \, d\alpha$.

Definition 2.2. A function $\gamma : [a, b] \to \mathbb{C}$ for $[a, b] \subseteq \mathbb{R}$ is of *bounded variation* if there is a constant $M > 0$ such that for any partition $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a, b]$

$$v(\gamma, P) = \sum_{k=1}^{|} \gamma(t_k) - \gamma(t_{k-1})| \leq M$$

The total variation of γ, $V(\gamma)$ is defined by

$$V(\gamma) = \sup_{P \in \mathcal{P}([a,b])} v(\gamma, P)$$

Proposition 2.3. *$\gamma : [a,b] \to \mathbb{C}$ is of bounded variation if and only if $\Re\gamma$ and $\Im\gamma$ are of bounded variation.*

Proof. Follows from the following inequality:

$$\max\{|u(t_k) - u(t_{k-1})|, |v(t_k) - v(t_{k-1})|\} \leq |\gamma(t_k) - \gamma(t_{k-1})| \leq |u(t_k) - u(t_{k-1})| + |v(t_k) - v(t_{k-1})|$$

■

Proposition 2.4. *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation. Then*

(a) If P and Q are partitions of $[a,b]$ with Q a refinement of P, then $v(\gamma, P) \leq v(\gamma, Q)$

(b) If $\sigma : [a,b] \to \mathbb{C}$ is also of bounded variation and $\alpha, \beta \in \mathbb{C}$ then $\alpha\gamma + \beta\sigma$ is of bounded variation and $V(\alpha\gamma + \beta\sigma) \leq |\alpha|V(\gamma) + |\beta|V(\sigma)$

Proof.

1. Let $[t_{i-1}, t_i]$ be an interval in the partition of P. Let $y \in Q \backslash P$ such that $y \in [t_{i-1}, t_i]$. Then, note that

$$|\gamma(t_i) - \gamma(t_{i-1})| \leq |\gamma(t_i) - \gamma(y)| + |\gamma(y) - \gamma(t_i)|$$

giving us the desired conclusion.

2. Similar to above, we have

$$|(\alpha\gamma + \beta\sigma)(t_i) - (\alpha\gamma + \beta\sigma)(t_{i-1})| \leq |\alpha||\gamma(t_i) - \gamma(t_{i-1})| + |\beta||\sigma(t_i) - \sigma(t_{i-1})|$$

Consequently,

$$v(\alpha\gamma + \beta\sigma, P) \leq |\alpha|v(\gamma, P) + |\beta|v(\sigma, P)$$

The conclusion is obvious.

■

Definition 2.5 (Smooth, Piecewise Smooth). A path in a region $G \subseteq \mathbb{C}$ is a continuous function $\gamma : [a,b] \to G$ for some $[a,b,] \subseteq \mathbb{R}$. If $\gamma'(t)$ exists for each $t \in [a,b]$ and $\gamma' : [a,b] \to \mathbb{C}$ is continuous, then γ issaid to be *smooth*. γ Is said to be *piecewise smooth* if there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$ such that γ is smooth on each subinterval $[t_{i-1}, t_i]$ for $1 \leq i \leq n$.

Proposition 2.6. *If $\gamma : [a,b] \to \mathbb{C}$ is piecewise smooth then γ is of bounded variation and*

$$V(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

Proof. We shall prove the statement in the case when γ is smooth on $[a, b]$. The general case follows from applying our proof to each piecewise smooth subinterval of $[a, b]$.

Let $a = t_0 < t_1 < \cdots < t_m = b$ be a partition, denoted by P. Then,

$$\begin{aligned} v(\gamma, P) &= \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\, dt \right| \\ &\leq \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} |\gamma'(t)|\, dt \\ &= \int_a^b |\gamma'(t)|\, dt \end{aligned}$$

First, this shows that γ is of bounded variation and further, $V(\gamma) \leq \int_a^b |\gamma'(t)|\, dt$. We shall show the reverse inequality, which would prove the theorem.

Let $\varepsilon > 0$. Since γ' is continuous on $[a, b]$, it must be uniformly continuous, therefore, there is $\delta > 0$ such that whenever $|s - t| < \delta$, we have $|\gamma'(s) - \gamma'(t)| < \varepsilon$.

Let $a = t_0 < t_1 < \cdots < t_m = b$ be a partition with mesh smaller than δ. Consequently, for all $1 \leq i \leq m$, we have for all $t \in [t_{i-1}, t_i]$,

$$|\gamma'(t) - \gamma'(t_i)| < \varepsilon \Longrightarrow |\gamma'(t)| < |\gamma'(t_i)| + \varepsilon$$

Hence,

$$\begin{aligned} \int_{t_{i-1}}^{t_i} |\gamma'(t)|\, dt &= |\gamma'(t_i)|\Delta t_i + \varepsilon \Delta t_i \\ &= \left| \int_{t_{i-1}}^{t_i} \gamma'(t_i) - \gamma'(t) + \gamma'(t)\, dt \right| + \varepsilon \Delta t_i \\ &\leq \left| \int_{t_{i-1}}^{t_i} \gamma'(t_i) - \gamma'(t)\, dt \right| + \left| \int_{t_{i-1}}^{t_i} \gamma'(t)\, dt \right| + \varepsilon \Delta t_i \\ &\leq \varepsilon \Delta t_i + |\gamma(t_i) - \gamma(t_{i-1})| + \varepsilon \Delta t_i \\ &= \big|\gamma(t_i) - \gamma(t_{i-1})\big| + 2\varepsilon \Delta t_i \end{aligned}$$

Adding together all these inequalities, we have

$$\int_a^b |\gamma'(t)|\, dt \leq v(\gamma, P) + 2\varepsilon(b - a) \leq V(\gamma) + 2\varepsilon(b - a)$$

Since ε was arbitrary, we have the desired conclusion. ■

Theorem 2.7. *Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \to \mathbb{C}$ is continuous. Then there is a complex number I such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that when P is a partition of $[a, b]$ with $\|P\| < \delta$, then*

$$\left| I - \sum_{k=1}^{m} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \varepsilon$$

for whatever choice of points $\tau_k \in [t_{k-1}, t_k]$.

This number I is called the *integral of f with respect to γ over $[a, b]$* and is designated by

$$I = \int f\, d\gamma$$

We first need the following lemma due to Cantor:

Lemma 2.8 (Cantor). *Let $A_1, A_2, \ldots$ be a sequence of non-empty compact, closed subsets of a topological space X such that $A_1 \supseteq A_2 \supseteq \cdots$. Then,*

$$\bigcap_{k=0}^{\infty} A_k \neq \varnothing$$

Proof. Suppose $\bigcap_{k=0}^{\infty} A_k = \varnothing$. Define $B_i = X \backslash A_i$, then, $\{B_i\}$ forms an open cover for A_1, consequently, has a finite subcover, say $\{B_{n_1}, \ldots, B_{n_k}\}$. Now, since

$$A_1 \subseteq \bigcup_{i=1}^{k} B_{n_i} \subseteq \bigcup_{j=1}^{n_k} B_j$$

This immediately implies that

$$A_{n_k} = A \cap \bigcap_{i=1}^{n_k} B_i = \varnothing$$

a contradiction. ∎

Proof of Theorem 2.7. Since f is continuous, it must be uniformly continuous. Thus, we can find positive numbers $\delta_1 > \delta_2 > \cdots$ such that if $|s-t| < \delta_m$, then $|f(s) - f(t)| < \frac{1}{m}$. Let $\mathscr{P}_m$ denote the colletion of all partitions P of $[a,b]$ with $\|P\| < \delta_m$. Note that we have $\mathscr{P}_1 \supseteq \mathscr{P}_2 \supseteq \cdots$. Finally define F_m to be the closure of

$$\left\{ S(P) := \sum_{k=1}^{n} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \mid P \in \mathscr{P}_m,\ t_{k-1} \leq \tau_k \leq t_k \right\} \tag{$\diamond$}$$

We shall show that the following hold:

$$\begin{cases} F_1 \supseteq F_2 \supseteq \cdots \\ \text{diam}\, F_m \leq \frac{2}{m} V(\gamma) \end{cases}$$

The first sequence of containments follows trivially from the definition of $\mathscr{P}_m$. Recall that in a metric space, diam $\overline{E}$ = diam E for all $E \subseteq X$. With this in mind, it suffices to show that the diameter of the set ($\diamond$) is at most $\frac{2}{m} V(\gamma)$.

We shall show that if $P \in \mathscr{P}_m$ and $P \subseteq Q$ are partitions of $[a,b]$, then $|S(P) - S(Q)| < \frac{1}{m} V(\gamma)$.

Choose any interval $[t_{k-1}, t_k]$ in the partition P and let Q refine it as

$$t_{k-1} = s_0 < s_1 < \cdots < s_n = t_k$$

Let $\chi_1, \ldots, \chi_n$ be a tagging of the refinement. Then,

$$\begin{aligned} &\left| f(\tau_k) \sum_{i=1}^{n} \gamma(s_i) - \gamma(s_{i-1}) - \sum_{i=1}^{n} f(\chi_i)(\gamma(s_i) - \gamma(s_{i-1})) \right| \\ &= \left| \sum_{i=1}^{n} (f(\tau_k) - f(\chi_i))(\gamma(s_i) - \gamma(s_{i-1})) \right| \\ &\leq \frac{1}{m} \sum_{i=1}^{n} |\gamma(s_i) - \gamma(s_{i-1})| \end{aligned}$$

Adding together these inequalities for each subinterval $[t_{k-1}, t_k]$, we have that $|S(P) - S(Q)| \leq \frac{1}{m} V(\gamma)$.

Let $P, R \in \mathscr{P}_m$ and Q be their common refinement. Then, we have

$$|S(P) - S(R)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma)$$

From this it follows that diam $F_m \leq \frac{2}{m}V(\gamma)$. Now, since diam $F_m \to 0$ as $m \to \infty$, it must be the case that $\bigcap_{m=1}^{\infty} F_m$ is a singleton set, containing a single complex number, say I.

Let $\varepsilon > 0$, choose $m > \frac{2}{\varepsilon}V(\gamma)$. Choose $\delta = \delta_m$. Since $I \in F_m$, it must be the case that $F_m \subseteq B(I, \varepsilon)$, giving us the desired conclusion. ■

Proposition 2.9. *Let $f, g : [a,b] \to \mathbb{C}$ be continuous functions and let $\gamma, \sigma : [a,b] \to \mathbb{C}$ be functions of bounded variation. Then for any scalars α and β,*

1. $\int_a^b \alpha f + \beta g \, d\gamma = \alpha \int_a^b f \, d\gamma + \beta \int_a^b g \, d\gamma$
2. $\int_a^b f \, d(\alpha\gamma + \beta\sigma) = \alpha \int_a^b f \, d\gamma + \beta \int_a^b f \, d\sigma$

Proof. ■

Lemma 2.10. *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : [a,b] \to \mathbb{C}$ be continuous. If $a = t_0 < t_1 < \cdots < t_n = b$ then*

$$\int_a^b f \, d\gamma = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f \, d\gamma$$

Theorem 2.11. *If γ is piecewise smooth and $f : [a,b] \to \mathbb{C}$ is continuous, then*

$$\int_a^b f \, d\gamma = \int_a^b f(t)\gamma'(t) \, dt$$

Proof. It suffices to consider the case where γ is smooth, since the general statement follows by applying our result to each piecewise smooth component and adding them up using Lemma 2.10.

We have that $\gamma = u + iv$ is smooth where $u, v : [a,b] \to \mathbb{R}$; thus, both u and v must be smooth, furthermore, $\gamma' = u' + iv'$. As a result, it suffices to prove the theorem for γ being real valued and smooth. We shall require the fact that is it real valued to apply the Mean Value Theorem.

Let $\varepsilon > 0$ and $\delta > 0$ be such that for any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$,

$$\left| \int_a^b f \, d\gamma - \sum_{k=1}^{n} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \frac{\varepsilon}{2}$$

$$\left| \int_a^b f(t)\gamma'(t) \, dt - \sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \frac{\varepsilon}{2}$$

for any choice of $\tau_k \in [t_{k-1}, t_k]$. Using the mean value theorem, choose τ_k such that

$$\gamma'(\tau_k) = \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}}$$

Consequently,

$$\left| \int_a^b f \, d\gamma - \int_a^b f(t)\gamma'(t) \, dt \right| < \varepsilon$$

and we have the desired conclusion. ■

Definition 2.12 (Bounded Variation). Let $\gamma : [a, b] \to \mathbb{C}$ be a path. The set $\{\gamma(t) \mid a \leq t \leq b\}$ is called the *trace of* γ and is denoted by $\{\gamma\}$. The path γ is said to be *rectifiable* if it is of bounded variation.

Definition 2.13 (Line Integral). If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and f is a function defined and continuous on the trace of γ. Then, the line integral of f along γ is

$$\int_a^b f(\gamma(t))\, d\gamma(t)$$

Theorem 2.14. *If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\varphi : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\varphi(c) = a$ and $\varphi(d) = b$. Then, for any function f continuous on $\{\gamma\}$,*

$$\int_\gamma f = \int_{\gamma \circ \varphi} f$$

Proof. Let $\varepsilon > 0$. Then, there is a δ_1 such that for all partitions $P = \{c = s_0 < s_1 < \cdots < s_n = d\}$ with $\|P\| < \delta$, and a tagging, $\sigma_k \in [s_{k-1}, s_k]$,

$$\left| \int_{\gamma \circ \varphi} f - \sum_{k=1}^{n} f(\gamma \circ \varphi(\sigma_k))(\gamma \circ \varphi(s_k) - \gamma \circ \varphi(s_{k-1})) \right| < \frac{\varepsilon}{2}$$

furthermore, whenever $s, t \in [c, d]$ with $|s - t| < \delta_1$, $|\varphi(s) - \varphi(t)| < \delta_2$ (note that we can do this since the function φ is uniformly continuous).

Choose $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ with $\|P\| < \delta_2$ and a tagging $\tau_k \in [t_{k-1} t_k]$, then

$$\left| \int_\gamma f - \sum_{k=1}^{n} f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) \right| < \frac{\varepsilon}{2}$$

Finally, let $\sigma_k = \varphi(\tau_k)$, then we have through a trivial manipulation that

$$\left| \int_\gamma f - \int_{\gamma \circ \varphi} f \right| < \varepsilon$$

■

Definition 2.15. Let $\sigma : [c, d] \to \mathbb{C}$ and $\gamma[a, b] \to \mathbb{C}$ be rectifiable paths. The path σ is *equivalent* to γ if there is a function $\varphi : [c, d] \to [a, b]$ which is continuous, strictly increasing, and with $\varphi(c) = a$ and $\varphi(d) = b$ such that $\sigma = \gamma \circ \varphi$.

A *curve* is an equivalence class of paths. A trace of a curve is the trace of any one of its members. A curve is smooth (piecewise smooth) if and only if some one of its representatives is smooth (piecewise smooth).

Definition 2.16. If γ is a rectifiable curve then denote by $-\gamma : [-b, -a] \to \mathbb{C}$ the curve defined by $(-\gamma)(t) = \gamma(-t)$ for $-b \leq t \leq -a$. This may also be denoted by γ^{-1} (although the former is more customary). For some $c \in \mathbb{C}$, let $\gamma + c : [a, b] \to \mathbb{C}$ denote the curve defined by $(\gamma + c)(t) = \gamma(t) + c$.

Definition 2.17. Let $\gamma[a,b] \to \mathbb{C}$ be a rectifiable path and for $a \le t \le b$, let $|\gamma|(t)$ be $V(\gamma, [a,t])$. That is,

$$|\gamma|(t) = \sup\left\{ \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| : \{a = t_0 < t_1 < \cdots < t_n = t\} \text{ is a partition of } [a,t] \right\}$$

Define

$$\int_\gamma f\,|dz| = \int_a^b f(\gamma(t))\,d|\gamma|(t)$$

Proposition 2.18. *Let γ be a rectifiable curve and suppose that f is a function continuous on $\{\gamma\}$. Then*

(a) $\int_\gamma f = -\int_{-\gamma} f$

(b) $\left|\int_\gamma f\right| \le \int_\gamma |f|\,|dz| \le V(\gamma)\sup\{|f(z)| : z \in \{\gamma\}\}$

(c) If $c \in C$, then $\int_\gamma f(z)\,dz = \int_{\gamma+c} f(z-c)\,dz$

Proof. All follow from definitions. ■

Theorem 2.19 (Fundamental Theorem of Calculus for Line Integrals). *Let G be open in $\mathbb{C}$ and let γ be a rectifiable path in G with initial and end points α and β respectively. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$, then*

$$\int_\gamma f = F(\beta) - F(\alpha)$$

We would require the following lemma in order to prove the above theorem

Lemma 2.20. *If G is an open set in $\mathbb{C}$, $\gamma : [a,b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$ and $|\int_\gamma f - \int_\Gamma f| < \varepsilon$.*

Proof. We shall divide the proof into two cases:

- Case I: G is an open disk, say $B(c,r)$

  Since $\{\gamma\}$ is compact, there is $\rho > 0$ such that $\{\gamma\} \subseteq \overline{B}(c,\rho) \subseteq G$. Consequently, we shall proceed with the assumption that $G = \overline{B}(c,\rho)$. Therefore, G is compact and f is uniformly continuous on G.

  Let $\varepsilon > 0$. Then, there is a δ_1 such that whenever $|s-t| < \delta_1$, $|f(s) - f(t)| < \varepsilon$. Similarly, there is $\delta_2 > 0$ such that whenever $|s-t| < \delta_2$, $|\gamma(s) - \gamma(t)| < \delta_1$.

  Furthermore, due to Theorem 2.7, there is a mesh size, δ_3 such that for any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ with $\|P\| < \delta_3$,

$$\left|\int_\gamma f - \sum_{k=1}^{n} f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right|$$

  Let $\delta = \min\{\delta_2, \delta_3\}$ and $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a,b]$ with $\|P\| < \delta$. Define the polygonal path Γ by

$$\Gamma(t) = \frac{1}{t_k - t_{k-1}}\left((t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)\right)$$

which is essentially the straight line joining the points $\gamma(t_{k-1})$ and $\gamma(t_k)$.

First, note that

$$\int_\Gamma f = \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t))\, dt$$

Then, we have

$$\begin{aligned}
\left| \int_\gamma f - \int_\Gamma f \right| &\leq \varepsilon + \left| \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) - \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t))\, dt \right| \\
&\leq \varepsilon + \left| \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\gamma(t_k)) - f(\Gamma(t))\, dt \right| \\
&\leq \varepsilon + \sum_{k=1}^n \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \left| \int_{t_{k-1}}^{t_k} f(\gamma(t_k)) - f(\Gamma(t))\, dt \right| \\
&\leq \varepsilon + \varepsilon \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| \leq \varepsilon(1 + V(\gamma))
\end{aligned}$$

This completes the proof for the first case.

- Case II: G is arbitrary

  Since $\{\gamma\}$ is compact, there is $r > 0$ such that for all $z \in \gamma$, $B(z, r) \subseteq G$. Using uniform continuity, there is $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$. Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition with $\|P\| < \delta$. Define $\gamma_k : [t_{k-1}, t_k] \to \mathbb{C}$. Note that $\{\gamma_k\} \subseteq B(\gamma(t_{k-1}), r)$ and thus, we can apply Case I to obtain a polygonal path Γ_k such that $|\int_{\gamma_k} f - \int_{\Gamma_k} f| < \varepsilon/n$. The conclusion is now obvious by pasting together all the Γ_k's.

■

Proof of Theorem 2.19. Again, we divide the proof into two cases:

- Case I: $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth.

  Then, we trivially have

$$\int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b F'(\gamma(t))\gamma'(t)\, dt = \int_a^b (f \circ \gamma)'(t)\, dt = F(\gamma(b)) - F(\gamma(a))$$

- Case II: General case

  Recall that a polygonal path is piecewise smooth. That is, for any polygonal path Γ that begins at $\gamma(a)$ and ends at $\gamma(b)$, $\int_\Gamma f = F(\gamma(b)) - F(\gamma(a))$. Since any rectifiable curve can be approximated by a polygonal path, we have a suitable Γ for every $\varepsilon > 0$ such that

$$\left| \int_\gamma f - (F(\beta) - F(\alpha)) \right| = \left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon$$

  giving us the desired conclusion.

■

Corollary 2.21. Let G, γ and f satisfy the same hypothesis as in Theorem 2.19. If γ is a closed curve, then

$$\int_\gamma f = 0$$

Recall that the fundamental theorem of calculus in real analysis claimed that each continuous function had a primitive. This is untrue in complex analysis. Consider the function $f(z) = |z|^2$. That is, $f(x+iy) = x^2 + y^2$. Suppose this has a primitive, say $F = U + iV$. Then, using **CR**, we must have

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = x^2 + y^2 \qquad \text{and} \qquad \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} = 0$$

This implies that $U(x, y) = u(x)$ for some function u, but this gives

$$u'(x) = x^2 + y^2$$

which is obviously not possible.

2.2 Power Series for Analytic Functions

Theorem 2.22 (Leibniz's Rule). *Let $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ be a continuous function and define $g : [c,d] \to \mathbb{C}$ yb*

$$g(t) = \int_a^b \varphi(s,t)\, ds$$

Then g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a,b] \times [c,d]$ then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s,t)\, ds$$

Proof. We shall first show that g is continuous. Since φ is continuous, it is uniformly continuous on $[a,b] \times [c,d]$. Choose some $t_0 \in [c,d]$. Then, there is a δ such that whenever $|(s,t) - (s',t')| < \delta$, $|\varphi(s,t) - \varphi(s',t')| < \varepsilon$. Consequently, whenever $|t - t_0| < \delta$, $|g(t) - g(t_0)| < (b-a)\varepsilon$. This implies continuity.

Fix a point $t_0 \in [c,d]$ and choose any $\varepsilon > 0$. Further, denote $\frac{\partial \varphi}{\partial t}$ by φ_2, which is given to be continuous, and thus, is uniformly continuous on $[a,b] \times [c,d]$. Let $\delta > 0$ be such that whenever $|(s,t) - (s',t')| < \delta$, $|\varphi_2(s',t') - \varphi(s,t)| < \varepsilon$. That is,

$$|\varphi_2(s,t) - \varphi_2(s,t_0)| < \varepsilon$$

whenever $|t - t_0| < \delta$ and $a \le s \le b$. Therefore, we have

$$\left| \int_{t_0}^{t} \varphi_2(s,\tau) - \varphi_2(s,t_0)\, d\tau \right| < \varepsilon |t - t_0|$$

Note that $\Phi(t) = \varphi(s,t) - t\varphi_2(s,t_0)$ is a primitive of $\varphi_2(s,t) - \varphi_2(s,t_0)$. Due to the fundamental theorem of calculus, we must have

$$\left|\varphi(s,t) - \varphi(s,t_0) - (t - t_0)\varphi_2(s,t_0)\right| \le \varepsilon |t - t_0|$$

for all $s \in [a,b]$ whenever $|t - t_0| < \delta$. This is equivalent to writing

$$-\varepsilon \ge \frac{\varphi(s,t) - \varphi(s,t_0)}{t - t_0} - \varphi_2(s,t_0) \le \varepsilon$$

Integrating both sides with respect to s, we have

$$\left| \frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \varphi_2(s,t_0)\, ds \right| \le \varepsilon(b-a)$$

This shows that g is differentiable and

$$g'(t) = \int_a^b \varphi_2(s,t)\, ds$$

Obviously the right hand side of the above equality is continuous and thus g is continuously differentiable. ■

Example 2.23. Let z be a complex number with $|z| < 1$. Then,

$$\int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\, ds$$

and equivalently stated, if $\gamma : [0, 2\pi] \to \mathbb{C}$ is a closed path given by $\gamma(t) = e^{it}$, then

$$\int_\gamma \frac{1}{x - z}\, dx = 2\pi$$

Proof. Define the function

$$g(t) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - tz}\, ds$$

for $0 \leq t \leq 1$. Note that in this region, the function

$$\varphi(s, t) = \frac{e^{is}}{e^{is} - tz}$$

is well defined, since $|e^{is}| = 1 > |tz|$.

Using Theorem 2.22, we have

$$g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2}\, ds$$

Consider the function

$$\Phi(s) = \frac{iz}{e^{is} - tz}$$

Notice that

$$\Phi'(s) = \frac{ze^{is}}{e^{is} - tz}$$

Then, using Theorem 2.19, $g'(t) = \Phi(2\pi) - \Phi(0) = 0$. Therefore, g is constant. The conclusion follows from calculating $t = 0$. ■

Proposition 2.24. *Let $f : G \to \mathbb{C}$ be analytic and suppose $\overline{B}(a, r) \subseteq G$ where $r > 0$. If $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, then*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

for $|z - a| < r$.

Proof. It is not hard to see that without loss of generality we may suppose that $a = 0$ and $r = 1$. Then, we would like to show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z}\, ds$$

for $|z| < 1$. This is equivalent to showing

$$\int_0^{2\pi} \left(\frac{f(e^{is})e^{is}}{e^{is} - z} - f(z) \right) ds = 0$$

Define the function

$$\varphi(s, t) = \frac{f(z + t(e^{is} - z))e^{is}}{e^{is} - z} - f(z)$$

and $g(t) = \int_0^{2\pi} \varphi(s,t)\, ds$. We would like to show that $g(1) = 0$.

Note that the function $\varphi(s,t)$ is well defined and continuously differentiable on the interval $[0, 2\pi] \times [0, 1]$ (it is here that we use the fact that $|z| < 1$). Then,

$$g'(t) = \int_0^{2\pi} f(z + t(e^{is} - z))e^{is}\, ds$$

Consider the function $\Phi(s) = \frac{1}{it} f(z + t(e^{is} - z))$. Trivially note that $\Phi'(s) = f(z + t(e^{is} - z))e^{is}$. Using the fundamental theorem of calculus, we have

$$g'(t) = \Phi(2\pi) - \Phi(0) = 0$$

Implying that g is constant on $[0, 1]$. Recall that we have already calculated

$$g(0) = \int_0^{2\pi} \frac{f(z)}{e^{is} - z} - f(z)\, ds = 0$$

This completes the proof. ■

Lemma 2.25. *Let γ be a rectifiable curve in $\mathbb{C}$ and suppose that F_n and F are continuous functions on $\{\gamma\}$ such that the sequence $\{F_n\}$ converges uniformly to F. Then*

$$\int_\gamma F = \lim_{n\to\infty} \int_\gamma F_n$$

Proof. Let $\varepsilon > 0$ be given. Then, there is a positive integer N such that for all $n \geq N$, $|F_n - F| \leq \varepsilon / V(\gamma)$. Then, we have (for all $n \geq N$)

$$\left| \int_\gamma F - F_n \right| \leq \int_\gamma |F - F_n|\, |dz| \leq \varepsilon$$

This completes the proof. ■

Theorem 2.26. *Let f be analytic in $B(a, R)$; then $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ for $|z - a| < R$, where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence $\geq R$.*

Proof. Let $z \in B(a, R)$. Choose $|z - a| < r < R$ and define γ to be the circle $\partial B(a, r)$. Then, using Proposition 2.24,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

Now, note that

$$\frac{1}{w - z} = \frac{1}{w - a} \cdot \frac{1}{1 - \frac{z-a}{w-a}} = \frac{1}{w - a} \sum_{k=0}^{\infty} \left(\frac{z - a}{w - a} \right)^k$$

Since $w \in \{\gamma\}$, there must exist $M > 0$ such that $|f(w)| < M$ for all $w \in \{\gamma\}$ and thus

$$\frac{|f(w)||z - a|^n}{|w - a|^{n+1}} \leq \frac{M}{r} \left(\frac{|z - a|}{r} \right)^n$$

Due to the Weierstrass M-test, the power series converges uniformly for $w \in \{\gamma\}$. And due to the Weierstrass M-test, the power series converges uniformly for $w \in \{\gamma\}$. Therefore, we may write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a} \sum_{k=0}^{\infty} \left(\frac{z-a}{w-a} \right) \\ &= \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, dw \right] (z-a)^n \end{aligned}$$

Define

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, dw$$

Then, the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges to $f(z)$ on $B(a,r)$. Consequently, f is infinitely differentiable at z and thus,

$$a_n = \frac{1}{n!} f^{(n)}(a)$$

Now, the characterization of a_n is independent of γ and therefore r. Consequently, this power series converges to $f(z)$ whenever $|z-a| < R$. Therefore, the radius of convergence must be at least R. ■

Corollary 2.27. If $f : G \to \mathbb{C}$ is analytic adn $a \in G$. Then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$ where $R = d(a, \partial G)$.

Corollary 2.28. If $f : G \to \mathbb{C}$ is analytic, then it is infinitely differentiable.

Corollary 2.29. If $f : G \to \mathbb{C}$ is analytic and $\overline{B}(a,r) \subseteq G$, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, dw$$

where $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$.

Proposition 2.30 (Cauchy's Estimate). *Let f be analtic in $B(a,R)$ and suppose $|f(z)| \leq M$ for all $z \in B(a,R)$. Then*

$$|f^{(n)}(a)| \leq \frac{n! M}{R^n}$$

Proof. Let $r < R$ and $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$.

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \left| \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, ds \right| \leq \int_\gamma \left| \frac{f(w)}{(w-a)^{n+1}} \right| |dw| \leq \frac{n! M}{r^n}$$

The result follows by letting $r \to R^-$. ■

Proposition 2.31. *Let f be analytic in the disk $B(a, R)$ and suppose that γ is a closed rectifiable curve in $B(a, R)$. Then*

$$\int_\gamma f = 0$$

Proof. It suffices to show that f has a primitive on $B(a, R)$ whence, we would be done by Theorem 2.19. Due to Theorem 2.26, there is a power series representation for f,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for $z \in B(a, R)$.

Define the function

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$$

Notice that the radius of convergence of F is equal to that of f and $F' = f$. As a result, F is a primitive for f on $B(a, R)$. ∎

2.3 Zeros of Analytic Functions

Definition 2.32 (Entire Function). An *entire function* si a function which isd efined and analytic in the whole complex plane $\mathbb{C}$.

We immediately obtain the following result:

Proposition 2.33. *If f is an entire function, then f has a power series expansion with infinite radius of convergence.*

Lemma 2.34. *No non-constant polynomial is bounded. That is, if $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in \mathbb{C}[z]$. Then, $\lim_{z\to\infty} p(z) = \infty$.*

Proof. Trivial. ∎

Theorem 2.35 (Liouville). *If f is a bounded entire function, then f is constant.*

In the proof of Liouville's Theorem, we shall require the following lemma:

Lemma 2.36. *If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$, then f is constant on G.*

Proof. Choose any $z_0 \in G$ and let $\omega_0 = f(z_0)$. Define $A = f^{-1}(\{z_0\})$. Obviously, A is closed in G. Choose $a \in A$ and $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq G$. Pick any $z \in B(a, \varepsilon)$ with $a \neq z$. Define $g(t) = f((1-t)a + tz)$. Note that $g'(s) = f'((1-t)a + tz)(z-a) = 0$, consequently, g is constant and therefore, $f(z) = g(1) = g(0) = \omega_0$. Therefore, $B(a, \varepsilon) \subseteq A$ and thus A is open. This shows that A must be equal to G, completing the proof. ∎

Prof of Theorem 2.35. Let $M > 0$ be such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Choose any $a \in A$. Then, for any $R > 0$, applying Proposition 2.30, we have

$$|f'(a)| \leq \frac{M}{R}$$

Letting $R \to \infty$, we have $f'(a) = 0$ for all $a \in \mathbb{C}$. We are now done due to the preceeding lemma. ■

We may now prove the fundamental theorem of algebra:

Theorem 2.37 (Fundamental Theorem of Algebra). *If $p(z)$ is a non-constant polyomial then there is a complex number a with $p(a) = 0$.*

Proof. Suppose not. Then, $f(z) = \frac{1}{p(z)}$ is entire. Since $\lim_{z\to\infty} p(z) = \infty$, $\lim_{z\to\infty} f(z) = 0$. Therefore, there is ε such that whenever $|z| > \varepsilon$, $|f(z)| < 1$. This immediately implies that f is bounded on $\mathbb{C}$, consequently is constant. A contradiction. ■

Let us look at another application of Liouville's Theorem.

Example 2.38. Let f be an entire function with $\Re(f)$ bounded above. Then, f is constant.

Proof. Consider the entire function $g(z) = \exp(f(z))$. Since $|g(z)| = |\exp(\Re(f(z)))|$, it is bounded and therefore, constant. Hence, $f(z)$ takes values in a discrete set and owing to it being a continuous map, it must be constant. ■

Theorem 2.39. *Let $G \subseteq \mathbb{C}$ be a region, and $f : G \to \mathbb{C}$ be an analytic function. Then the following are equivalent*

(a) $f \equiv 0$

(b) there is a point $a \in G$ succh that $f^{(n)}(a) = 0$ for each $n \geq 0$

(c) the set $f^{-1}(\{0\})$ has a limit point in G

Proof. It is clear that $(a) \implies (b) \wedge (c)$. We shall show that $(c) \implies (b)$ and $(b) \implies (a)$.

- $\underline{(c) \implies (b)}$: Let a be a limit point of the set $f^{-1}(\{0\})$. We shall show that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}_0$. Let n be the smallest integer ≥ 1 such that $f^{(r)}(a) = 0$ for all $r < n$. Now, there is $R > 0$ such that $B(a, R) \subseteq G$, and thus there is a power series expansion around a for all $z \in B(a, R)$, given by

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k$$

Define the function

$$g(z) = \sum_{k=0}^{\infty} a_{n+k} (z-a)^k$$

Then $g(a) = a_n \neq 0$. It is not hard to see that $g(z)$ is analytic in $B(a, R)$, as a result, there is some $0 < r < R$ such that $g(z) \neq 0$ for each $z \in B(a, r)$. But since a is a limit point of the set $f^{-1}(\{0\})$, there is some $b \neq a$ in $f^{-1}(\{0\}) \cap B(a, r)$, and we have $0 = f(b) = (b-a)^n g(b)$, a contradiction. This shows that no such $n \in \mathbb{N}$ can exist.

- $\underline{(c) \implies (b)}$: Let $A = \{z \in G \mid f^{(n)}(z) = 0,\ \forall\, n \in \mathbb{N}\}$. We shall show that A is clopen in G. Indeed, let $a \in A$. Since G is open, there is $R > 0$ such that $B(a, R) \subseteq G$. Let $b \in B(a, R)$. Note that f has a power series expansion around a that is valid for all $z \in B(a, R)$. Since $a \in A$, this power series expansion is identically zero, as a result, $f(b) = 0$ and $B(a, R) \subseteq A$ and A is open.

  Next, let $\{z_k\}$ be a sequence of points in A converging to $a \in G$. Then, using continuity of $f^{(n)}$, we conclude that $f^{(n)}(a) = \lim f^{(n)}(z_k) = 0$ and A is closed. This completes the proof.

■

Lemma 2.40. *Let $G \subseteq \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Then f is constant.*

Proof. ■

Theorem 2.41 (Maximum Modulus Theorem). *Let $G \subseteq \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ be an analytic function such that there is $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$. Then f is constant on G.*

Proof. Let $r > 0$ be such that $B(a, r) \subseteq G$ and let γ be the curve given by $\gamma(t) = a + re^{it}$. Then, we have

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - a}\, dw \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) \end{aligned}$$

and equivalently,

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})|\, dt \leq |f(a)|$$

As a result,

$$\int_0^{2\pi} |f(a)| - |f(a + re^{it})|\, dt = 0$$

since the integrand is a continuous nonnegative function of t, it must be identically zero. As a result, f maps the ball $B(a, r)$ to the circle $|z| = |f(a)|$. Due to Lemma 2.40, f is constant on $B(a, r)$. Since $B(a, r)$ has at least one limit point in G (say a for example), it must be constant on G. ■

2.4 Cauchy's Theorem

Definition 2.42 (Homotopy for Closed Curves). Let $G \subseteq \mathbb{C}$ and $\gamma_0, \gamma_1 : [0, 1] \to G$ be two closed rectifiable curves. Then γ_0 is *homotopic* to γ_1 in G if there is a continuous function *Gamma* : $[0, 1] \times [0, 1] \to G$ such that

$$\begin{cases} \Gamma(s, 0) = \gamma_0(s) \text{ and } \gamma(s, 1) = \gamma_1(s) & 0 \leq s \leq 1 \\ \Gamma(0, t) = \Gamma(1, t) & 0 \leq t \leq 1 \end{cases}$$

We denote this by $\gamma_0 \simeq \gamma_1 \pmod{G}$.

Lemma 2.43. *The relation $\simeq$ is an equivalence relation over the set of all closed curves in G.*

Proof. Standard proof from Algebraic Topology. ■

Theorem 2.44 (Cauchy). *Let $G \subseteq \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ be analytic. Let γ_0 and γ_1 be homotopic closed curves. Then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Proof. Let $\Gamma : I^2 \to G$ be the homotopy taking γ_0 to γ_1. Since I^2 is compact, so is $\Gamma(I^2)$. Consequently, due to the Lebesgue Number Lemma, there is $r > 0$ such that for all $a \in \Gamma(I^2)$, $B(a,r) \subseteq G$. Using the uniform continuity of Γ, there is $\delta > 0$ such that whenever $|(s',t') - (s,t)| < \delta$, $|\Gamma(s',t') - \Gamma(s,t)| < r$. Choose $n \in \mathbb{N}$ such that $\sqrt{2}/n < \delta$. Finally, let γ_t denote the curve $\Gamma(s,t)$ where t is fixed and $0 \le s \le 1$.

Let $Z_{i,j}$ denote the point $\Gamma\left(\frac{i}{n}, \frac{j}{n}\right)$ and $Q_{i,j}$ denote the square $\left(\frac{i}{n}, \frac{j}{n}\right) \to \left(\frac{i+1}{n}, \frac{j}{n}\right) \to \left(\frac{i+1}{n}, \frac{j+1}{n}\right) \to \left(\frac{i}{n}, \frac{j+1}{n}\right) \to \left(\frac{i}{n}, \frac{j}{n}\right)$. We shall show that

$$\int_{\Gamma(Q_{i,j})} f = 0$$

which would imply the desired conclusion through a straightforward inductive process.

But since $|z_1 - z_2| < \sqrt{2}/n < \delta$ for all $z_1, z_2 \in Q_{i,j}$, we can conclude that $\Gamma(Q_{i,j}) \subseteq B\left(Z_{i,j}, r\right)$, whence we are done due to Proposition 2.31. ■

Corollary 2.45. Let $G \subseteq \mathbb{C}$ be a region and γ a closed rectifiable curve in G which is nulhomotopic. Then,

$$\int_{\gamma} f = 0$$

for every analytic function f defined on G.

Corollary 2.46. Let $G \subseteq \mathbb{C}$ be a region and γ_0, γ_1 be path homotopic curves. Then,

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

for every analytic function f defined on G.

Corollary 2.47. If $G \subseteq \mathbb{C}$ is simply connected then $\int_{\gamma} f = 0$ for every closed rectifiable curve $\gamma \subseteq G$ and every analytic function $f : G \to \mathbb{C}$.

Theorem 2.48. *If G is simply connected and $f : G \to \mathbb{C}$ is analytic in G, then f has a primitive in G.*

Proof. Fix some basepoint $a \in G$ and for each $z \in G$, define $F : G \to \mathbb{C}$ as $F(z) = \int_{\gamma} f$. Due to the previous result, this function is well defined. We shall show that F is a primitive for f on G. Let $z_0 \in G$. Since G is open, there is $r > 0$ such that $\overline{B}(z_0, r) \subseteq G$. Note that this is a convex set centered at z_0, as a result, all line segments between two points are contained in it. Choose some $z \in B(z_0, r)$. Then,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0, z]} (f(w) - f(z_0))\, dw$$
$$\implies \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \left| \frac{1}{z - z_0} \right| \int_{[z_0, z]} \left| (f(w) - f(z_0)) \right| \, |dw|$$

Let $\varepsilon > 0$ be given. Note that $\overline{B}(z_0, r)$ is compact in G and thus, f is uniformly continuous. As a result, there is a small enough $r > 0$ such that for all $z \in B(z_0, r)$, $|f(z) - f(z_0)| < \varepsilon$. And thus,

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \varepsilon$$

which implies the desired conclusion. ■

Theorem 2.49 (Morera). *Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ be a continuous function. If for every triangular path Δ in G, the value of $\int_\Delta f = 0$, then f is analytic over G.*

Proof. Note that it suffices to show this in the case $G = B(a, R)$ for some $a \in \mathbb{C}$ and $R > 0$, since for every $a \in G$, there is an open ball containing it and showing the analyticity of f every such ball would imply the analyticity of f on G.

Let $[x, y]$ denote the straight line segment that begins at x and ends at y. Define the function $F : G \to \mathbb{C}$ by

$$F(z) = \int_{[a,z]} f$$

We shall show that $F' = f$, which would imply the analyticity of F and therefore that of f. Choose some $z_0 \in G$. For any $z \in G$, we have

$$F(z) - F(z_0) = \int_{[a,z]} f - \int_{[a,z_0]} f = \int_{[z_0,z]} f$$

Then,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f - f(z_0))$$

Choose $r > 0$ such that $\overline{B}(z_0, r) \subseteq G$. Since f is continuous on G, it is uniformly continuous on $\overline{B}(z_0, r)$. Let $\varepsilon > 0$ be given. There is $\delta > 0$ such that whenever $|z - z_0| < \delta$, $|f(z) - f(z_0)| < \varepsilon$. Consequently, for all such z, we have

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \frac{1}{|z - z_0|} \int_{[z_0,z]} \left| f(t) - f(z_0) \right| \, |dt| \leq \varepsilon$$

This completes the proof. ■

Theorem 2.50 (Goursat). *Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ be differentiable. Then, f is analytic over G.*

Proof. Due to Morera's Theorem, it suffices to show that for every triangular path $\Delta = [a, b, c, a] \subseteq G$, the value $\int_\Delta f = 0$.

We shall define a sequence of closed triangular regions $\Delta = \Delta^{(0)} \supseteq \Delta^{(1)} \supseteq \cdots$. Obviously, since each triangular region is closed and bounded, it must be compact.

Divide the triangle $\Delta^{(i)}$ into four congruent triangles using the midpoint of each side. Let the smaller triangles be denoted by $\Delta_1, \ldots, \Delta_4$. Define

$$j = \operatorname{argmax}_{j \in \{1,\ldots,4\}} \left| \int_{\Delta_j} f \right| \qquad \text{and} \qquad \Delta^{(i+1)} = \Delta_j$$

We have

$$\begin{cases} \left| \int_{\Delta^{(i)}} f \right| \leq 4 \left| \int_{\Delta^{(i+1)}} f \right| \\ 2 \operatorname{diam} \Delta^{(i+1)} = \operatorname{diam} \Delta^{(i)} \\ 2V(\Delta^{(i+1)}) = V(\Delta^{(i)}) \end{cases}$$

Then, using Lemma 2.8, $\bigcap_{i=0}^{\infty} \Delta^{(i)}$ is singleton, say $\{z_0\}$. Choose some $\varepsilon > 0$. Since f is differentiable at z_0, there is $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $|z - z_0| < \delta$. Choose $n \in \mathbb{N}$ such that $\operatorname{diam} \Delta^{(n)} = \frac{1}{2^n} \operatorname{diam} \Delta < \delta$. Therefore, $\Delta^{(n)} \subseteq B(z_0, \delta)$. Then, we have

$$\int_{\Delta^{(n)}} f = \int_{\Delta^{(n)}} f(z) - f(z_0) - (z - z_0) f'(z_0)\, dz$$

whence

$$\begin{aligned} \left| \int_{\Delta^{(n)}} f \right| &= \left| \int_{\Delta^{(n)}} f(z) - f(z_0) - (z - z_0) f'(z_0)\, dz \right| \\ &\leq \int_{\Delta^{(n)}} |f(z) - f(z_0) - (z - z_0) f'(z_0)|\, |dz| \\ &\leq \int_{\Delta^{(n)}} \varepsilon |z - z_0|\, |dz| \\ &\leq \varepsilon \operatorname{diam} \Delta^{(n)} V(\Delta^{(n)}) \\ &= \varepsilon (\operatorname{diam} \Delta) V(\Delta) \frac{1}{4^n} \end{aligned}$$

from which it follows that

$$\left| \int_\Delta f \right| \leq 4^n \left| \int_{\Delta^{(n)}} f \right| \leq \varepsilon (\operatorname{diam} \Delta) V(\Delta)$$

Since ε was arbitrary, we have the desired conclusion. ■

Due to Theorem 2.50, we may redefine an analytic function in its more accepted definition.

Definition 2.51 (Analytic). Let $G \subseteq \mathbb{C}$ be open. Then $f : G \to \mathbb{C}$ is said to be analytic if it is differentiable over G.

2.5 Winding Numbers

Proposition 2.52. *If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

is an integer.

Proof. The proof is divided into two parts. First, we prove the statement of the proposition for all piecewise smooth curves.

- Case I: γ is piecewise smooth
- Case II: γ is an arbitrary rectifiable curve

■

Definition 2.53 (Winding Number). If γ is a closed rectifiable curve in $\mathbb{C}$ then for $a \notin \{\gamma\}$,

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$$

is called the *winding number* of γ around a.

Theorem 2.54 (Cauchy's Integral Formula). *Let $f : G \to \mathbb{C}$ be analytic and $\gamma \subseteq G$ be a nulhomotopic rectifiable closed contour. Then, for $a \notin \{\gamma\}$,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a} = n(\gamma; a) f(a)$$

Proof. Note that the function $f(z) - f(a)$ is analytic and has a zero at $z = a$, therefore, there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) - f(a) = g(z)(z-a)$. From here, we have that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a} = \frac{1}{2\pi i}\int_\gamma g(z) = 0$$

and therefore,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a} = \frac{1}{2\pi i}\int_\gamma \frac{f(a)}{z-a} = n(\gamma; a) f(a)$$

where the last equality follows from the definition of the winding number. ■

Lemma 2.55. *Let $G \subseteq \mathbb{C}$ be a region and $\gamma \subseteq G$ be a closed rectifiable contour and $\varphi : \{\gamma\} \to \mathbb{C}$ be continuous. For each positive integer m, let*

$$F_m(z) = \int_\gamma \frac{\varphi(w)}{(w-z)^m}\,dw$$

Then F_m is analytic on $\mathbb{C}\backslash\{\gamma\}$. Furthermore, $F_m'(z) = mF_{m+1}(z)$.

Proof. Fix some $a \in \mathbb{C} \backslash \{\gamma\}$. Now, there is $R > 0$ such that $B(a, R) \subseteq \mathbb{C} \backslash \{\gamma\}$. Consider some $z \in B(a, R)$. Then,

$$
\begin{aligned}
F_m(z) - F_m(a) &= \frac{1}{2\pi i} \int_\gamma \varphi(w) \left[\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} \right] dw \\
&= \frac{1}{2\pi i} \int_\gamma \varphi(w) \left(\frac{1}{w-z} - \frac{1}{w-a} \right) \left(\sum_{k=0}^{m-1} \frac{1}{(w-z)^k (w-a)^{m-k-1}} \right) dw \\
&= \frac{z-a}{2\pi i} \int_\gamma \varphi(w) \left(\sum_{k=1}^{m} \frac{1}{(w-z)^k (w-a)^{m+1-k}} \right) dw
\end{aligned}
$$

From here, it follows that

$$\frac{F_m(z) - F_m(a)}{z-a} = \frac{1}{2\pi i} \int_\gamma \varphi(w) \left(\sum_{k=1}^{m} \frac{1}{(w-z)^k (w-a)^{m+1-k}} \right) dw$$

in the limit $z \to a$, we get

$$F_m'(z) = \frac{m}{2\pi i} \int_\gamma \frac{\varphi(w)}{(w-a)^m} \, dw = mF_{m+1}(z)$$

It is now easy to see that the function is analytic. ■

Theorem 2.56 (Extended Cauchy's Integral Formula). *Let $f : G \to \mathbb{C}$ be an analytic function and $\gamma \subseteq G$ be a closed contour of bounded variation. Then, for every $a \in G \backslash \{\gamma\}$, and every nonnegative integer n,*

$$n(\gamma; a) f^{(n)}(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, dw$$

Proof. Follows from the above lemma. ■

2.6 The Open Mapping Theorem

Theorem 2.57. *Let $G \subseteq \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ be analytic having zeros $a_1, \ldots, a_n$ counting multiplicity in G. Then, for any closed curve $\gamma \subseteq G$, we have*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} = \sum_{k=1}^{n} n(\gamma; a_k)$$

Proof. Recall that if f has a zero at $z = a$, then there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = (z-a)g(z)$. Continuing this way, we have an analytic function $h : G \to \mathbb{C}$ such that $f(z) = \prod_{k=1}^{n}(z-a)h(z)$. Then,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} = \frac{1}{2\pi i} \int_\gamma \sum_{k=1}^{n} \frac{1}{z-a_k} + \frac{h'(z)}{h(z)}$$

Since the function h has no zeros in G, the function h'/h is analytic on G and therefore, the integral is 0. The conclusion now follows. ■

Lemma 2.58. *Let f be analytic on $B(a, R)$ for some $R > 0$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then there is an $\varepsilon > 0$ and $\delta > 0$ such that for $0 < |\zeta - \alpha| < \delta$, the equation $f(z) = \zeta$ has exactly m simple roots in $B(a, \varepsilon)$.*

Proof. ■

In particular, if $m \geq 1$, then for each $\zeta \in B(\alpha, \delta)$, there is a corresponding $\xi \in B(a, \varepsilon)$ such that $f(\xi) = \zeta$. Therefore, $B(\alpha, \delta) \subseteq f(B(a, \varepsilon))$.

Theorem 2.59 (Open Mapping Theorem). *Let $G \subseteq \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ be analytic. Let U be open in G. Then $f(U)$ is open in $\mathbb{C}$.*

Proof. Choose some $a \in U$. Then, there is some $R > 0$ such that $B(a, R) \subseteq U$. Due to Theorem 2.57 and the remark following it, there is $\varepsilon > 0$ and $\delta > 0$ such that $B(f(a), \delta) \subseteq f(B(a, \varepsilon))$. The conclusion is immediate now. ■

Corollary 2.60. Suppose $f : G \to \mathbb{C}$ is one-one, analytic and $f(G) = \Omega$. Then $f^{-1} : \Omega \to \mathbb{C}$ is analytic and $(f^{-1})'(\omega) = f'(z)^{-1}$ where $\omega = f(z)$.

Proof. From Theorem 2.59, it is immediate that f is a homeomorphism. Let $g = f^{-1}$. We have $g \circ f = \mathbf{i}d$, from which the conclusion follows. ■

2.7 The Complex Logarithm

In this section, we shall construct the complex logarithm, which is an inverse function to the analytic function $\exp : \mathbb{C} \to \mathbb{C}$. In particular, we shall prove a more general theorem, which would immediately imply the existence of the complex logarithm.

Theorem 2.61. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected region and $f : \Omega \to \mathbb{C}$ be an analytic function which does not vanish on Ω. Then, there is an analytic function $g : \Omega \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in \Omega$.*

Proof. Fix a basepoint $z_0 \in \Omega$ and define

$$g(z) = \int_\gamma \frac{f'(z)}{f(z)}\, dz + c_0$$

where γ is any path from z_0 to z and $c_0 \in \mathbb{C}$ is such that $e^{c_0} = f(z_0)$, which exists since f does not vanish on Ω. Further, since f'/f is analytic on Ω, the function g is analytic on Ω.

Consider the analytic function $h = fe^{-g}$ on Ω. Differentiating this function, we have

$$h'(z) = f'(z)e^{-g(z)} - f(z)g'(z)e^{-g(z)} = 0$$

whence h is constant on Ω. Since $h(z_0) = 1$, we are done. ■

Note that the domain being simply connected is essential lest there be an analytic function $g : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ such that $e^{g(z)} = 1/z$, which is a contradiction, since the integral of the former over the unit circle is zero while the integral of the latter is $2\pi i$.

With the above theorem in hand, we may define arbitrary powers of an analytic function on a simply connected region. Indeed, let $\alpha \in \mathbb{R}$, then, we may define

$$z^\alpha = e^{\alpha \log z}$$

where log is a branch of the logarithm in the aforementioned simply connected region.

Chapter 3

Singularities and Residue Calculus

3.1 Classification of Singularities

Definition 3.1. A function f has an *isolated singularity* at a point $z = a$ if there is $R > 0$ such that f is analytic on $0 < |z - a| < R$. The point a is called a *removable singularity* if there is an analytic function $g : B(a, R) \to \mathbb{C}$ such that $f(z) = g(z)$ for $0 < |z - a| < R$.

Theorem 3.2. *If f has an isolated singularity at a, then the point $z = a$ is a removable singularity if and only if*

$$\lim_{z \to a}(z - a)f(z) = 0$$

Proof. The forward direction is obvious. We shall show the reverse direction, that is, suppose $\lim_{z\to a}(z - a)f(z) = 0$. There is $R > 0$ such that f is analytic in $0 < |z - a| < R$. Now, define the function $g : B(a, R) \to \mathbb{C}$ such that $g(z) = (z - a)f(z)$. It is obvious that g is continuous. It suffices to show that g is analytic, since then, there would exist an analytic function h such that $g(z) = (z - a)h(z)$, implying the desired conclusion.

To show that g is analytic, we shall use Morera's Theorem. Let T be a triangle in $B(a, R)$. Note that since this region is convex, it suffices to choose any three points a, b, c in the interior and they would form a valid triangle. Let Δ denote the interior of T. If $a \notin \Delta$, then T is nulhomotopic and due to Theorem 2.44, the integral $\int_T g$ must be zero.

Next, if a is a vertex of the triangle, say $[a, b, c, a]$, then for any points x and y on the line segments $[a, b]$ and $[a, c]$,

$$\int_{[a,b,c,a]} g = \int_{[a,x,y]} g + \int_{[x,b,c,y]} g = \int_{[a,x,y]} g$$

where the last equality follows from Theorem 2.44. Since g is continuous, there is $r > 0$ such that for all $t \in B(a, r)$, $|g(t)| < \varepsilon$. And thus, $|\int_{[a,x,y]} g| < \varepsilon\ell$ where ℓ is the permieter of T. It is now obvious that the integral must be zero.

Finally, suppose $a \in \Delta$ where $T = [b, c, d, b]$. The integral is now given by

$$\int_{[b,c,d,a]} g = \int_{[a,b,c,a]} g + \int_{[a,c,d,a]} g + \int_{[a,d,b,a]} g = 0$$

This completes the proof. ∎

Definition 3.3 (Pole, Essential Singularity). If $z = a$ is an isolated singularity of f, then a is a *pole* of f if $\lim_{z\to a} |f(z)| = \infty$. If an isolated singularity is niether a pole nor a removable singularity, it is then called an *essential singularity*.

Theorem 3.4. *Let $f : G\setminus\{a\} \to \mathbb{C}$ be analytic with a pole at $z = a$. Then there is an analytic function $g : G \to \mathbb{C}$ and a positive integer m such that*

$$f(z) = \frac{g(z)}{(z-a)^m} \quad \text{on } G\setminus\{a\}$$

and $g(a) \neq 0$.

Proof. Consider the analytic function $h : G\setminus\{a\} \to \mathbb{C}$ given by $h = \frac{1}{f}$. Then it is obvious that $\lim_{z\to a} f(z) = 0$, as a result, f has a removable singularity at $z = a$, and thus, there is an analytic function $\tilde{h} : G \to \mathbb{C}$ such that $h = \tilde{h}$ on G. Now, since $\tilde{h}(a) = 0$, there is a positive integer m and an analytic function $g : G \to \mathbb{C}$ such that $\tilde{h}(z) = (z-a)^m g(z)$. As a result, we see that

$$f(z) = \frac{1}{(z-a)^m}\frac{1}{g(z)}$$

and the conclusion follows. ■

Definition 3.5. If f has a pole at $z = a$, and m is the smallest positive integer such that $f(z)(z-a)^m$ has a removable singularity at $z = a$, then f is said to have a *pole of order m* at $z = a$.

Definition 3.6. Let $\{z_n\}_{n\in\mathbb{Z}}$ be a doubly infinite sequence of complex numbers. We say that $\sum_{n=-\infty}^{\infty} z_n$ is *absolutely convergent* if both $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ are absolutely convergent.

We denote the annular region $R_1 < |z-a| < R_2$ by $\operatorname{ann}(a, R_1, R_2)$.

Theorem 3.7 (Laurent Series Development). *Let f be analytic on $\operatorname{ann}(a, R_1, R_2)$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where the convergence is absolute and uniform over $\overline{\operatorname{ann}}(a, r_1, r_2)$ for $R_1 < r_1 < r_2 < R_2$. Also the coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

where γ is the circle $|z-a| = r$ for all $R_1 < r < R_2$. Furthermore, this series is unique.

Proof. ■

3.2 Residues

Definition 3.8. Let f have an isolated singularity at $z = a$ and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

be its Laurent expansion about $z = a$. Then the *residue* of f at $z = a$ is defined as a_{-1}.

Theorem 3.9 (Weak Residue Theorem). *Let f be analytic in the region G except for isolated* **poles** *$a_1, \ldots, a_n \in G$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if γ is nulhomotopic in G, then*

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{n} n(\gamma, a_k)\operatorname{Res}(f, a_k)$$

Proof. Let S_j denote the singular part of f at a_j. Then, $g = f - \sum_{k=1}^n S_k$ has removable singularities at $a_1, \ldots, a_n$. As a result,

$$0 = \int_\gamma g = \int_\gamma f - \sum_{k=1}^{n}\int_\gamma S_k$$

and the conclusion follows. ■

There is a stronger version of the above theorem wherein the word *poles* is replaced by *singularities*. We shall prove this later.

Proposition 3.10. *Suppose f has a pole of order m at $z = a$ and let $g(z) = (z-a)^m f(z)$. Then,*

$$\operatorname{Res}(f, a) = \frac{1}{(m-1)!} g^{(m-1)}(a)$$

Proof. Follows from the definition. ■

Evaluating Integrals using the Residue Theorem

Example 3.11. Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$$

Solution. Define the contour

$$\gamma := [-R, R] \cup \underbrace{\{Re^{it} \mid t \in [0, \pi]\}}_{\Gamma} \qquad R > 1$$

and the function $f(z) = \frac{z^2}{1+z^4}$, which has poles of order 1 at

$$\operatorname{cis}\left(\frac{\pi}{4}\right), \operatorname{cis}\left(\frac{3\pi}{4}\right), \operatorname{cis}\left(\frac{5\pi}{4}\right), \operatorname{cis}\left(\frac{7\pi}{4}\right)$$

Within our contour, we have only $a_1 = \operatorname{cis}\left(\frac{\pi}{4}\right)$ and $a_2 = \operatorname{cis}\left(\frac{3\pi}{4}\right)$ and

$$\operatorname{Res}(f, a_1) = \lim_{z \to a_1}(z - a_1)f(z) = \frac{1}{4a_1} = \frac{1}{4}\operatorname{cis}\left(-\frac{\pi}{4}\right)$$
$$\operatorname{Res}(f, a_2) = \lim_{z \to a_2}(z - a_2)f(z) = \frac{1}{4a_2} = \frac{1}{4}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

$$\int_\gamma f(z)\,dz = \frac{\pi i}{2}\left(\text{cis}\left(-\frac{\pi}{4}\right) + \text{cis}\left(-\frac{3\pi}{4}\right)\right) = \frac{\pi}{\sqrt{2}}$$

Now,

$$0 \le \int_\Gamma f \le \int_\Gamma \frac{R^2}{|1+z^4|}\,|dz| \le \int_\Gamma \frac{\pi R^3}{R^4-1}$$

And in the limit $R \to \infty$, $\int_\Gamma f = 0$. The conclusion follows. ∎

Example 3.12. Show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin \pi a}$$

for $0 < a < 1$.

Proof. Consider the function $f(z) = \frac{e^{az}}{1+e^z}$, which is analytic except for poles at $(2k+1)\pi i$ for all $k \in \mathbb{Z}$. Let γ denote the rectangular contour:

$$-R \longrightarrow R \longrightarrow R + 2\pi i \longrightarrow -R + 2\pi i \longrightarrow -R$$

We note that

$$n(\gamma, (2k+1)\pi i) = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore,

$$\lim_{z \to \pi i} (z - \pi i)\frac{e^{az}}{1+e^z} = -e^{a\pi i}$$

Therefore, we have, due to Theorem 3.9, that

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{az}}{1+e^z}\,dz = -e^{a\pi i}$$

It is not hard to argue that the integral on the segments $R \to R + 2\pi i$ and $-R + 2\pi i \to -R$ both tend to 0 as $R \to \infty$. Thus, in the limit $R \to \infty$, we have

$$\int_{-R}^{R} f + \int_{R+2\pi i}^{-R+2\pi i} f = -e^{a\pi i}$$

Further,

$$\int_{R+2\pi i}^{-R+2\pi i} f = e^{2a\pi i}\int_R^{-R} \frac{e^{ax}}{1+e^x}dx$$

Thus,

$$(1 - e^{2a\pi i})\int_{-\infty}^{\infty} f = (-2\pi i)e^{a\pi i}$$

Thus,

$$\int_{-\infty}^{\infty} f = \frac{2\pi i}{e^{a\pi i} - e^{-a\pi i}} = \frac{\pi}{\sin \pi a}$$

∎

The next example has a rather unmotivated solution but we present it anyways since it is an important result to keep in mind.

Example 3.13. Let $u \in \mathbb{R}\backslash\mathbb{Z}$. Then, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi}{\sin^2 \pi u}$$

Proof. Consider the meromorphic function

$$f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$$

It has poles at k for $k \in \mathbb{Z}$ and $-u$. Let N be an integer such that $N > |u|$ and let $R = N + 1/2$. This contour contains the following poles:

$$\{-u\} \cup \{k \in \mathbb{Z} \mid -N \leq k \leq N\}$$

The residue at $z = k \in \mathbb{Z}$ is given by

$$\lim_{z \to k} (z-k) \frac{\pi \cot \pi z}{(u+z)^2} = \frac{\pi}{(u+k)^2}$$

On the other hand, the residue at $z = -u$ is the coefficient a_{-1} in the Laurent expansion of $f(z)$ around $z = -u$. Since u is not an integer, $\pi \cot \pi z$ is analytic in a ball around u, and the required coefficient is given by $f'(u) = -\frac{\pi^2}{\sin^2 \pi u}$. Hence,

$$\sum_{n=-N}^{N} \frac{\pi}{(u+n)^2} = \int_{|z|=R} f(z)\, dz + \frac{\pi^2}{\sin^2 \pi u}$$

Therefore, it suffices to show that the integral on the circle is zero. TODO: Add in later ■

3.3 Argument Principle

Definition 3.14 (Meromorphic). A function which is analytic on a region except for poles is said to be *meromorphic* on that region.

Theorem 3.15 (Argument Principle). *Let f be meromorphic in G with poles $p_1, \ldots, p_m$ and zeros $z_1, \ldots, z_n$ counted according to multiplicity. If γ is a closed rectifiable curve which is nulhomotopic and not passing through any of the aforementioned points, then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{n} n(\gamma, z_k) - \sum_{k=1}^{m} n(\gamma, p_k)$$

Proof. It is not hard to argue that there is an analytic function g on G that does not vanish anywhere such that

$$\frac{f'}{f} = \sum_{k=1}^{n} \frac{1}{z - z_k} - \sum_{k=1}^{m} \frac{1}{z - p_k} + \frac{g'}{g}$$

Note that g'/g is an analytic function and due to Cauchy's Theorem,

$$\int_\gamma \frac{f'}{f} = \sum_{k=1}^{n} n(\gamma, z_k) - \sum_{k=1}^{m} n(\gamma, p_k)$$

This completes the proof. ■

Corollary 3.16. Let f be meromorphic in G with poles $p_1, \ldots, p_m$ and zeros $z_1, \ldots, z_n$ counted according to multiplicity. If γ is a closed rectifiable curve which is nulhomotopic and not passing through any of the aforementioned points, then for an analytic function g on G,

$$\frac{1}{2\pi i}\int_\gamma g\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} g(z_k)n(\gamma, z_k) - \sum_{k=1}^{m} g(p_k)n(\gamma, p_k)$$

Theorem 3.17 (Rouché). *Suppose f and g are meromorphic in the region G and $\overline{B}(a,R) \subseteq G$. If f and g have no zeros or poles on the circle $\gamma := \{z : |z-a| = R\}$ and $|f(z) - g(z)| < |g(z)|$ on γ, then*

$$Z_f - P_f = Z_g - P_g$$

where Z_f, Z_g denote the zeros of f and g in $B(a,R)$ and P_f, P_g denote the poles of f and g in $B(a,R)$.

First Proof. First, note that

$$\left|1 - \frac{f(z)}{g(z)}\right| < 1$$

for all $z \in \{\gamma\}$. Since $(f/g)(\{\gamma\}) \subseteq B(1,1)$, there is a neighborhood of $\{\gamma\}$ that is mapped into $B(1,1)$. As a result, on this neighborhood, $\log(f/g)$, the principal branch is a primitive for $(f/g)'/(f/g)$. As a result, we have

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{(f/g)} = \frac{1}{2\pi i}\int_\gamma \left(\frac{f'}{f} - \frac{g'}{g}\right)$$

The conclusion follows. ∎

Proof 2. Define the function $h_t(z) = tf(z) + (1-t)g(z)$ for all $t \in [0,1]$. Then, $h_0(z) = g(z)$ and $h_1(z) = f(z)$, further, note that on γ,

$$|h_t(z)| = |g(z) + t(f(z) - g(z))| > 0.$$

Let

$$n_t = \frac{1}{2\pi i}\int_\gamma \frac{h_t'(z)}{h(z)}\,dz$$

Then, n_t is obviously an integer. We contend that the map $t \mapsto n_t$ is continuous. Indeed, $h_t'(z)/h_t(z)$ is a joint continuous function of t and z since both the numerator and denominator are continuous in t and z, and the denominator does not vanish on γ as we have argued above.

Now, since n_t only takes integral values, it must be a constant function of t and the conclusion follows. ∎

We now give an alternate proof of the open mapping theorem using Theorem 3.17

Alternate proof of Theorem 2.59. ∎ Add proof

3.4 Runge's Theorem

This section is taken from [Con78].

Theorem 3.18. *Let $K \subseteq \mathbb{C}$ be compact and $E \subseteq \mathbb{C}_\infty \backslash K$ which meets every component of $\mathbb{C}_\infty \backslash K$. If f is analytic in an open set Ω containing K and $\varepsilon > 0$, then there is a rational function $R(z)$ with poles only in E such that*

$$|f(z) - R(z)| < \varepsilon$$

for all $z \in K$.

We prove this result through a series of lemmas. The setup is as mentioned in the statement of Theorem 3.18 and shall not be repeated.

Lemma 3.19. *There are straight line segments $\gamma_1, \ldots, \gamma_n$ in $\Omega \backslash K$ such that*

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} \, dw$$

for all $z \in K$. The line segments form a finite number of closed polygons.

Chapter 4

Conformal Maps

Definition 4.1 (Conformal Map). A *conformal map* is a bijective holomorphis function $f : U \to V$ where U and V are open sets in $\mathbb{C}$. In this case, U and V are said to be conformally equivalent.

We have seen, as a corollary to Theorem 2.59, that a bijective holomorphic function has a holomorphic inverse. That is, $f^{-1} : V \to U$ is also conformal.

Example 4.2. Define $\mathbb{H}$ to be the upper half plane, that is, the set of complex numbers with positive imaginary part. We contend that $\mathbb{H}$ is conformally equivalent to $\mathbb{D}$, the unit disk. Consider the map $F : \mathbb{D} \to \mathbb{H}$ given by

$$F(z) = i\frac{1-z}{1+z}$$

Indeed, for $z = u + iv$, we have

$$\begin{aligned}\operatorname{Im}(F(z)) &= \operatorname{Re}\left(\frac{1-u-iv}{1+u+iv}\right) \\ &= \frac{1-u^2-v^2}{(1+u)^2+v^2} > 0\end{aligned}$$

Define the map $G : \mathbb{H} \to \mathbb{D}$ given by

$$F(z) = \frac{i-z}{i+z}$$

It is not hard to see that $F \circ G = \mathbf{id}_{\mathbb{H}}$ and $G \circ F = \mathbf{id}_{\mathbb{D}}$. This completes the proof.

4.1 Schwarz Lemma and applications

Lemma 4.3 (Schwarz). *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then,*

(a) $|f(z)| \le |z|$ for all $z \in \mathbb{D}$.

(b) if for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$, then f is a rotation.

(c) $|f'(0)| \le 1$ and if equality holds, then f is a rotation.

Proof. The function $f(z)/z$ has a removable singularity at 0, and consequently is holomorphic on $\mathbb{D}$. Pick some $0 < r < 1$. Then, for all $|z| = r$, we have

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$$

Then, due to the maximum modulus principle, $|f(z)/z| \leq 1/r$ for all $z \in \mathbb{D}$ whence (a) follows.

As for (b), we would have $|f(z_0)/z_0| = 1$ for some $z_0 \in \mathbb{D}\backslash\{0\}$, and due to the maximum modulus principle, $f(z)/z$ must be constant, and the conclusion follows.

Finally, for (c), note that $g(0) = f'(0)$, consequently, if $g(0) = 1$, then due to the maximum modulus principle, g is constant, thereby completing the proof. ■

Proposition 4.4. *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. If f is non-constant, then it has atmost one fixed point.*

Proof. ■

4.1.1 Automorphisms of $\mathbb{D}$ and $\mathbb{H}$

Throughout this section, an *automorophism* of a domain U refers to a conformal map $f : U \to U$.

Disk

First, we shall study the automorphisms of $\mathbb{D}$. Pick some $\alpha \in \mathbb{D}$ and consider the map $\psi_\alpha : \mathbb{D} \to \mathbb{D}$ given by

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}.$$

Notice that both maps $z \mapsto \alpha - z$ and $z \mapsto 1 - \overline{\alpha}z$ are holomorphic and since $|\alpha| < 1$, their quotient is also holomorphic on $\mathbb{D}$. Finally, for any $z \in \mathbb{D}$,

$$\begin{aligned}
|\psi_\alpha(z)|^2 &= \left|\frac{\alpha - z}{1 - \overline{\alpha}z}\right|^2 \\
&= \frac{\overline{\alpha}\alpha + \overline{z}z - \overline{\alpha}z - \overline{z}\alpha}{1 - \overline{\alpha}z - \overline{z}\alpha + \overline{\alpha}\alpha\overline{z}z} \\
&= 1 - \frac{(1 - \overline{\alpha}\alpha)(1 - \overline{z}z)}{1 - \overline{\alpha}z - \overline{z}\alpha + \overline{\alpha}\alpha\overline{z}z} < 1
\end{aligned}$$

whence ψ_α is a biholomorphic map from $\mathbb{D}$ to $\mathbb{D}$. These are called the **"Blaschke Factors"**. These are automorphisms of order two, that is, $\psi_\alpha \circ \psi_\alpha = \mathbf{id}_{\mathbb{D}}$.

Theorem 4.5. *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic automorphism. Then there is $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta}\psi_\alpha(z)$$

Proof. Since f is bijective, there is a unique $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. Define $g = f \circ \psi_\alpha$. Then $g : \mathbb{D} \to \mathbb{D}$ is a biholomorphic map such that $g(0) = 0$. We shall show that g is a rotation. Let $h : \mathbb{D} \to \mathbb{D}$ be the inverse of g, which is also biholomorphic. We have due to Lemma 4.3, that $|g(z)| \leq |z|$ and $|h(z)| \leq |z|$ for all $z \in \mathbb{D}$. Putting these two together, we have

$$|z| = |h \circ g(z)| \leq |g(z)| \leq |z| \qquad \forall\, z \in \mathbb{D}$$

Thus, $|g(z)| = |z|$ for all $z \in \mathbb{D}$, whence, due to Lemma 4.3, g is a rotation and the proof is complete. ■

4.1.2 Upper Half Plane

4.2 The Riemann Mapping Theorem

Theorem 4.6 (Riemann). *Suppose $\Omega \subseteq \mathbb{C}$ is open and simply connected. Given $z_0 \in \Omega$, there is a unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.*

4.2.1 Montel's Theorem

Definition 4.7. Let $G \subseteq \mathbb{C}$ be open. A family $\mathcal{F}$ of holomorphic functions on G is said to be *normal* if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of G.

The family $\mathcal{F}$ is said to be *uniformly bounded on compact subsets of* G if for each compact set $K \subseteq G$, there is $M > 0$ such that $|f(z)| \leq M$ for all $z \in K$ and $f \in \mathcal{F}$.

The family $\mathcal{F}$ is said to be *equicontinuous* on a compact set $K \subseteq G$, for every $\varepsilon > 0$, there is $\delta > 0$ such that whenever $w, z \in K$ with $|z - w| < \delta$, $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$.

Note that there is a more general definition of equicontinuity, but in the case of a compact metric space, it is equivalent to the above.

Theorem 4.8 (Montel). *Suppose $\mathcal{F} \subseteq H(\mathbb{C})$ is a family of holomorphic functions on $G \subseteq \mathbb{C}$ that is uniformly bounded on compact subsets of G. Then,*

(a) $\mathcal{F}$ is equicontinuous on every compact subset of G

(b) $\mathcal{F}$ is a normal family

Note that (b) is a consequence of the Arzelà-Ascoli Theorem from topology, a proof of which can be found in this document.

Definition 4.9. A sequence $\{K_\ell\}_{\ell=1}^{\infty}$ of compact subsets of G is said to be an *exhaustion* if

(a) K_ℓ is contained in the interior of $K_{\ell+1}$ for all $\ell \in \mathbb{N}$

(b) Any compact set $K \subseteq G$ is contained in K_ℓ for some ℓ. In particular,

$$G = \bigcup_{\ell=1}^{\infty} K_\ell$$

Lemma 4.10. *Any open set $G \subseteq \mathbb{C}$ has an exhaustion.*

Proof. ■

Proof of Theorem 4.8. (a) Ket $K \subseteq G$ be compact. Now, there is $\delta > 0$ such that for all $z \in K$, $B(z, \delta) \subseteq G$. Let $r = \delta/3$. For $a, b \in K$ with $|a - b| < r$, we have

$$f(a) - f(b) = \frac{1}{2\pi i} \int_{|z-a|=2r} f(z) \left(\frac{1}{z-a} - \frac{1}{z-b} \right) dz$$

Consequently, we have

$$|f(a) - f(b)| \leq \frac{1}{2\pi} \int_{|z-a|=2r} |f(z)| \frac{|a-b|}{|z-a||z-b|} \, |dz|$$

We now use the inequality $|z-b| \geq r$ and $|a-b| \leq r$, which gives us

$$|f(a) - f(b)| \leq \frac{1}{2\pi} \cdot 4\pi r \cdot \frac{M|a-b|}{2r^2} = \frac{M|a-b|}{r}$$

Since this inequality holds for every $f \in \mathcal{F}$, we have equicontinuity.

(b) Let $\{K_n\}_{n=1}^{\infty}$ be an exhaustion of G and $\{f_n\}_{n=1}^{\infty}$ a sequence of functions in $\mathcal{F}$. We now work inductively by repeatedly applying Arzelà's theorem.

First, there is a subsequence $\{g_{n,1}\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ that converges uniformly on K_1. From this subsequence, extract $\{g_{n,2}\}_{n=1}^{\infty}$ that converges uniformly on K_2 and continue in this fashion. It is not hard to show that $\{g_{n,n}\}_{n=1}^{\infty}$ converges uniformly on every compact subset of G. This completes the proof. ■

Proposition 4.11. *Let $G \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n=1}^{\infty}$ a sequence of holomorphic functions that converge uniformly on every compact subset of G to the function $f : G \to \mathbb{C}$. Then f is holomorphic. Further, if each $\{f_n\}$ is injective, then f is either injective or constant.*

Proof. The holomorphicity of f follows from Theorem 2.49. We shall show that f is injective. Suppose there are two distinct $z_1, z_2 \in G$ such that $f(z_1) = f(z_2)$. Define the function $g : G \to \mathbb{C}$ by $g(z) = f(z) - f(z_1)$. Then, define the sequence of functions $\{g_n\}_{n=1}^{\infty}$ by $g_n(z) = f_n(z) - f_n(z_1)$. Obviously, g_n converges to g uniformly on every compact subset of G. If g is not identically zero, then there z_2 is an isolated zero, due to the Identity Theorem. Therefore, we may choose a circle γ centered at z_2 such that the only zero of g in the interior of γ is z_2.

Then, we have

$$1 = \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)}\, dz$$

Since g does not vanish on γ, and $g_n \to g$ uniformly on γ, we must have that $1/g_n \to 1/g$ uniformly on γ. Further, $g_n' \to g'$ uniformly on γ. Therefore,

$$\frac{1}{2\pi i} \int_\gamma \frac{g_n'(z)}{g(z)}\, dz \to \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)}\, dz$$

but this is absurd since every integral on the left is zero. This completes the proof. ■

4.2.2 Proof of the Riemann Mapping Theorem

Step I. We shall show that Ω is conformally equivalent to an open subset of $\mathbb{D}$.

Since Ω is a proper subset of $\mathbb{C}$, there is some $\alpha \in \mathbb{C} \backslash \Omega$. Define the holomorphic function $f(z) = \log(z - \alpha)$, which makes sense since $z - \alpha$ never vanishes on Ω.

Now, pick some point $w \in \Omega$. We contend that $f(w) + 2\pi i$ is contained in an open disk that is disjoint from $f(\Omega)$. For if not, then there is a sequence $\{z_n\}_{n=1}^{\infty}$ of points in Ω that converge to $f(w) + 2\pi i$. Since e^z is a continuous function, we see that z_n must converge to w, which would imply that $f(z_n)$ converges to $f(w)$, a contradiction.

Now, consider the map $F : \Omega \to \mathbb{C}$

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}$$

First, for each $z \in \Omega$, since $|f(z) - (f(w) + 2\pi i)|$ is bounded from below, $|F(z)|$ is bounded. Further, since f is injective, so is F. By translation and scaling of F, since it is bounded, we may embed Ω into $\mathbb{D}$.

Step II. In this step we shall construct our candidate for the required biholomorphic map.

Now, we may suppose without loss of generality that Ω is a domain contained in $\mathbb{D}$. We shall now construct a conformal map from Ω to $\mathbb{D}$. Define

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid f \text{ is holomorphic, injective and } f(0) = 0\}$$

Obviously, $\mathcal{F}$ is nonempty, since it contains the identity map and by construction, $\mathcal{F}$ is uniformly bounded. Due to Proposition 2.30, we see that $|f'(0)|$ must also be bounded for every $f \in \mathcal{F}$.

Let $s = \sup_{f \in \mathcal{F}} |f'(0)|$

Step III. We shall show that our chosen candidate $f : \Omega \to \mathbb{D}$ is in fact a biholomorphic map.

According to our construction, f is injective. It suffices to show that it is surjective. Suppose not and there is $\alpha \in \mathbb{D}$ which is not in the image of f. Let ψ_α be the Blaschke factor and consider the composition $\psi_\alpha \circ f : \Omega \to \mathbb{D}$. This is a holomorphic injective function whose image does not contain the origin. Let $U = (\psi_\alpha \circ f)(\Omega)$. Since U is open, simply connected (owing to it being a biholomorphic image of Ω) and does not contain the origin, we may define a complex logarithm on U, whence by composing, we can define a holomorphic function $g : U \to \mathbb{C}$ given by

$$g(z) = e^{\frac{1}{2} \log z}$$

Now, consider the function

Chapter 5

Series and Product Developments

Lemma 5.1. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions converging uniformly on every compact subset of K to a function $f : \Omega \to \mathbb{C}$. Then f is holomorphic and the sequence $\{f_n'\}$ converges uniformly on every compact subset of Ω to f'.*

5.1 Weierstrass' Theorem

Theorem 5.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on an open set $\Omega \subseteq \mathbb{C}$. If there is a sequence of positive constants $\{c_n\}$ such that*

$$\sum_{n=1}^{\infty} c_n < \infty \quad \text{and} \quad |f_n(z) - 1| \leq c_n \ \forall z \in \Omega$$

then

(a) The product $\prod_{n=1}^{\infty} f_n(z)$ converges uniformly in Ω to a holomorphic function $F : \Omega \to \mathbb{C}$.

(b) If $f_n(z)$ does not vanish for any n, then

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{f_n'(z)}$$

Proof. Define $a_n : \Omega \to \mathbb{C}$ given by $a_n(z) = f_n(z) - 1$. First, by disregarding finitely many terms of the product, we may suppose without loss of generality that $c_n < 1/2$. For any $N \in \mathbb{N}$, we have

$$\prod_{n=1}^{N} (1 + a_n(z)) = \prod_{n=1}^{n} e^{\log(1+a_n(z))} = e^{\sum_{n=1}^{N} \log(1+a_n(z))}.$$

In the above manipulation, we take log to be the principal branch of the logarithm which makes sense since for all $z \in \Omega$, $1 + a_n(z) \in B(1, 1/2) \subseteq \mathbb{C} \backslash \{z \in \mathbb{R} \mid z < 0\}$.

Now, it is not hard to see, using the power series expansion of the principal branch of log that $|\log(1 + z)| \leq 2|z|$ if $|z| < 1/2$, and thus $|\log(1 + a_n(z))| \leq 2|a_n(z)| \leq 2c_n$ on Ω. Let $b_N(z) = \sum_{n=1}^{N} \log(1 + a_n(z))$. Since $|b_n(z)|$ is bounded on Ω and converges uniformly to some analytic function $b : \Omega \to \mathbb{C}$.

The sequence e^{b_n} converges pointwise to e^b but since this is a uniformly bounded sequence, the convergence is uniform and e^b is analytic. This proves (a).

Define $G_n(z) = \prod_{k=1}^{n} f_k(z)$. Then, using the product rule,

$$\frac{G_n'(z)}{G_n(z)} = \sum_{k=1}^{n} \frac{f_n'(z)}{f_n(z)}.$$

We have shown in part (a) that G_n converges uniformly to F. Let $K \subseteq \Omega$ be a compact subset. Due to Lemma 5.1, G_n' converges uniformly to F' on K. Further, since $1/G_n$ is uniformly bounded, above on K[1] and thus G_n'/G_n converges to F'/F uniformly on K which finishes the proof. ■

Definition 5.3. Define the entire maps $E_n : \mathbb{C} \to \mathbb{C}$ for $n \geq 0$ by

$$E_0(z) = 1 - z \qquad E_n(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right) \text{ for } n \geq 1$$

These are called the *elementary factors*.

Theorem 5.4 (Weierstrass). *Given any sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers with $|a_n| \to \infty$ as $n \to \infty$, there exists an entire function f vanishing at exactly $\{a_n\}_{n=1}^{\infty}$ and nowhere else. Any other such entire function is of the form $f(z)e^{g(z)}$ where g is entire.*

First, we prove the second part of the theorem. Let f_1, f_2 be two entire functions satisfying the statement of the theorem. Then, f_2/f_1 has removable singularities at each a_n whence is entire. Using this entire function,

[1]This is because each f_n does not vanish on K and eventually, $|f_n| < 1$ since the sum of c_n's converges.

TODO List

1. Complete proof of Riemann Mapping Theorem
2. Complete the write up of Runge's Theorem
3. After Runge's Theorem, Mittag-Leffler
4. Phragmén-Lindelöf Theorem
5. Weierstrass Product Theorem
6. Hadamard Product Theorem

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