Group Transfer

Swayam Chube

December 1, 2024

§1 Introduction

Definition 1.1 (The Transfer Map). Let G be a group and $H \leqslant G$ be a subgroup of finite index, say n. Let $t_1, \dots, t_n$ be a left traversal for H in G. For every $g \in G$, and $1 \leqslant i \leqslant n$,

$$gt_i = t_{j_i} h_i$$

for some $1 \leqslant j_i \leqslant n$ and $h_i \in H$. Define

$$\psi(g) = \prod_{i=1}^{n} h_i \pmod{H'}$$

This defines a map $\psi : G \to H^{ab}$ called the *transfer*.

Proposition 1.2. The map ψ is independent of the choice of coset traversal of H in G.

Proof. Gadha mehnat. ∎

Theorem 1.3. Let $T = \{t_1, \dots, t_n\}$ be a left traversal for H in G. Then, for each $g \in G$, there is a subset $T_0 \subseteq T$ and positive integers n_t for each $t \in T_0$ such that

(a) $\sum_{t \in T_0} n_t = n$.

(b) $t^{-1} g^{n_t} t \in H$ for all $t \in T_0$.

(c) $\psi(g) = \prod_{t \in T_0} t^{-1} g^{n_t} t \pmod{H'}$.

(d) If g has finite order, then each n_t divides $|g|$.

Proof. The group $\langle g \rangle$ acts on T by left multiplication and decomposes T into orbits. Let T_0 be a set of representatives of these orbits. For each $t \in T_0$, let n_t denote the size of the orbit containing t. Then, note that

$$g^{n_t} t = tH.$$

There is some $h_t \in H$ such that $h_t = t^{-1}g^{n_t}t$. It follows that

$$\psi(g) = \prod_{t \in T_0} t^{-1}g^{n_t}t \pmod{H'},$$

which proves all four parts of the theorem. ∎

COROLLARY. If H is central and of finite index in G, then the transfer map $\psi : G \to H^{ab}$ is given by $g \mapsto g^n \pmod{H'}$ where $n = [G : H]$.

COROLLARY. If H is of finite index in G such that no two elements of H are conjugate in G, then the restriction of the transfer map $\psi|_H$ is given by $h \mapsto h^n$ where $n = [G : H]$.

§2 SOME APPLICATIONS

PROPOSITION 2.1. Let $A \trianglelefteq G$ be abelian of finite index and $\psi : G \to A$ the transfer map.

(a) $\psi(G) \subseteq Z(G)$.

(b) If G is finite and A is a Hall subgroup of G, then $\psi(G) = \psi(A) = A \cap Z(G)$. In this case, $G \cong \psi(G) \times \ker \psi$.

Proof. (a) Let $t_1, \ldots, t_n$ be a left traversal for A in G and choose $a \in G$. Let $t_{j_i}H = at_iH$. Let $g \in G$ be arbitrary. We know

$$\psi(a) = \prod_{i=1}^{n} t_{j_i}^{-1}at_i.$$

Then,

$$g^{-1}\psi(a)g = \prod_{i=1}^{n}(t_{j_i}g)^{-1}a(t_ig).$$

Since A is normal, it follows that $\{t_ig \mid 1 \leqslant i \leqslant n\}$ is a left traversal for A in G. This shows that $g^{-1}\psi(a)g = \psi(a)$ whence, $\psi(a)$ is central in G.

(b) From (a), it follows that $\psi(A) \subseteq \psi(G) \subseteq A \cap Z(G)$. Note that the restriction of ψ to $A \cap Z(G)$ is $a \mapsto a^n$ where $n = [G : A]$. Since A is a Hall subgroup, n is coprime to $|A|$, hence, to $|A \cap Z(G)|$. Consequently, the restriction of ψ to $A \cap Z(G)$ is an automorphism. It now follows that $A \cap Z(G) \subseteq \psi(A)$.

Finally, consider the exact sequence

$$1 \to \ker \psi \to G \xrightarrow{\psi} A \cap Z(G) \to 1.$$

This splits on the right and the splitting is central. Hence, $G \cong \ker \psi \times \psi(G)$. This completes the proof.

∎

THEOREM 2.2 (SCHUR). Let $[G : Z(G)] < \infty$. Then, G', the commutator subgroup, is a finite subgroup of G.

Proof. Let $g_1, \ldots, g_n$ be a left traversal for $Z(G)$ in G. Then, G' is generated by $\{[g_i, g_j] \mid 1 \leqslant i, j \leqslant n\}$, that is, G' is finitely generated. Further, the transfer map $\psi : G \to Z(G)$ is given by $\psi(g) = g^n$. Since $Z(G)$ is abelian, $G' \subseteq \ker \psi$. Hence, every element of G' is killed by n.

Consider $H = G' \cap Z(G)$. This is a finite index abelian subgroup of G', hence, is finitely generated. Further, it is killed by n, whence it is finite. It follows that G' is finite. ■

PROPOSITION 2.3. Let $S \subseteq G$ be the set of elements of finite order in G. If S is finite, then it is a subgroup of G.

Proof. Replace G by the subgroup generated by S. It suffices to show that G is finite, since then it would follow that $G = S$. Being the intersection of finitely many groups of finite index, we can conclude that $H = \bigcap_{s \in S} C_G(s)$ has finite index in G. But $H \subseteq Z(G)$ and hence, $[G : Z(G)] < \infty$, consequently, $|G'| < \infty$. Finally, note that G^{ab} is a finitely generated torsion abelian group, hence, finite. This shows that G is finite, thereby completing the proof. ■

PROPOSITION 2.4. Let G be a finite group of square free order.

§3 BURNSIDE'S COMPLEMENT THEOREM

DEFINITION 3.1 (FOCAL SUBGROUP). Let $H \leqslant G$ be a subgroup. Then the *focal subgroup* of H in G is defined as

$$\mathrm{Foc}_G(H) = \langle x^{-1}y \mid x, y \in H, \text{ and are } G-\text{conjugate} \rangle.$$

THEOREM 3.2. Let G be finite, $H \leqslant G$ a Hall subgroup and $\psi : G \to H^{ab}$ the transfer map. Then,

$$\mathrm{Foc}_G(H) = H \cap G' = H \cap \ker \psi.$$

Proof. If $y = gxg^{-1}$ for some $g \in G$, then $x^{-1}y = [x^{-1}, g] \in G'$ and hence, $\mathrm{Foc}_G(H) \subseteq H \cap G'$. On the other hand, $G' \subseteq \ker \psi$ since ψ is a homomorphism to an abelian group. Apriori, we have the following inclusions

$$\mathrm{Foc}_G(H) \subseteq H \cap G' \subseteq H \cap \ker \psi.$$

Let $g \in H \cap \ker \psi$. It suffices to show that $g \in \mathrm{Foc}_G(H)$. Using Theorem 1.3,

$$\psi(g) = \prod_{t \in T_0} t^{-1} g^{n_t} t \pmod{H'} = g^n \prod_{t \in T_0} g^{-n_t} t^{-1} g^{n_t} t \pmod{H'}.$$

According to Theorem 1.3, we also know that $t^{-1}g^{n_t}t \in H$ and hence, each factor $g^{-n_t}t^{-1}g^{n_t}t \in \mathrm{Foc}_G(H)$.

But since $g \in \ker(\psi)$, we must have that the product $\psi(g)$ as an element of H, lies in $H' \subseteq \mathrm{Foc}_G(H)$. But since each factor $g^{-n_t}t^{-1}g^{n_t}t \in \mathrm{Foc}_G(H)$, we must have $g^n \in \mathrm{Foc}_G(H)$. Recall that H is a Hall subgroup and hence, n is relatively prime to $|H|$, consequently, relatively prime to $|g|$. As a result, $g \in \langle g^n \rangle \subseteq \mathrm{Foc}_G(H)$. This completes the proof. ■

LEMMA 3.3 (BURNSIDE). Let P be a p-Sylow subgroup of G and suppose $x, y \in C_G(P)$ are conjugate in G. Then x and y are conjugate in $N_G(P)$.

Proof. Suppose $y = x^g$ for some $g \in G$. Then, $P \subseteq C_G(y) \cap C_G(x)$. Consequently,

$$P^g \subseteq C_G(x)^g = C_G(x^g) = C_G(y).$$

Since both P and P^g are Sylow p-subgroups of $C_G(y)$, there is a $c \in C_G(y)$ such that $P^{cg} = P$. Therefore, $cg \in N_G(P)$, and

$$x^{cg} = (x^g)^c = y^c = y.$$

This completes the proof. ■

DEFINITION 3.4. A group G is said to have a *normal p-complement* if there is a normal subgroup $N \trianglelefteq G$ such that $[G : N] = p^n$ where $n = v_p(|G|)$.

THEOREM 3.5 (BURNSIDE). Let P be a Sylow p-subgroup of G and suppose $P \subseteq Z(N_G(P))$. Then, G has a normal p-complement.

Proof. We contend that $\text{Foc}_G(P) = 1$. Indeed, suppose $x, y \in P$ are conjugate in G. According to our assumption on P, $P \subseteq C_G(P)$, therefore, there is some $g \in N_G(P)$ such that $y = gxg^{-1}$. But since $P \subseteq Z(N_G(P))$, we must have $y = x$ and hence, $\text{Foc}_G(H) = 1$. Using Theorem 3.2, we see that $P \cap \ker \psi = 1$ where $\psi : G \to P^{ab} = P$ is the transfer map. Therefore, $|\psi(P)| = |P|$, whence ψ is surjective. This shows that $\ker \psi$ is a normal p-complement in G. ■

THEOREM 3.6. Let G be a finite group such that every Sylow subgroup of G is cyclic. Then G is solvable.

Proof. Let p be the smallest prime dividing the order of G and P be a Sylow p-subgroup. Due to the N/C-theorem, there is an injection $N_G(P)/C_G(P) \hookrightarrow \text{Aut}(P)$. If $|P| = p^r$, then $\text{Aut}(P)$ has order $p^{r-1}(p-1)$. But since both $N_G(P)$ and $C_G(P)$ contain P, the order of the quotient $N_G(P)/C_G(P)$ cannot be divisible by p, hence, must be 1. Thus, $P \subseteq Z(N_G(P))$. Due to Theorem 3.5, G has a normal p-complement, say N.

This fits into a short exact sequence

$$1 \to N \to G \to G/N \to 1,$$

where G/N is a p-group, hence, solvable and $N \subsetneq G$ is a proper subgroup divisible by one less prime and hence, solvable due to an inductive argument. This completes the proof. ■