The Quillen-Suslin Theorem

Swayam Chube

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§1 Finite Free Resolutions

Definition 1.1. A module E is said to be *stably free* if there exists a finite free module F of such that $E \oplus F$ is finite free.

E is said to have a *finite free resolution* if there is a resolution

$$0 \to E_n \to \cdots \to E_0 \to E \to 0$$

such that each E_i is a finite free module.

Proposition 1.2. Let M be projective. Then M is stably free if and only if M admits a finite free resolution.

Proof. Suppose first that M is stably free. Then, there is a finite free F such that $E = M \oplus F$ is finite free. Thus, $0 \to F \to E \to M \to 0$ is a finite free resolution of M.

On the other hand, suppose M admits a finite free resolution,

$$0 \to E_n \to \cdots \to E_0 \to M \to 0,$$

where n is the smallest such. We shall induct on this n. The base case with $n = 0$ is trivial since M is free. Let $M_1 = \ker(E_0 \to M)$. Then, M_1 has a finite free resolution

$$0 \to E_n \to \cdots \to E_1 \to M_1 \to 0$$

of length $n - 1$ whence the induction hypothesis applies and there is a finite free F such that $M_1 \oplus F$ is finite free. Using the fact that M is projective, we have

$$M \oplus (M_1 \oplus F) \cong (M \oplus M_1) \oplus F \cong E \oplus F,$$

and hence, M is stably free. ■

Definition 1.3. A resolution

$$0 \to E_n \to \cdots \to E_0 \to M \to 0$$

is said to be *stably free* if each E_i is stably free for $0 \leqslant i \leqslant n$.

Proposition 1.4. M has a finite free resolution of length $n \geqslant 1$ if and only if it has a stably free resolution of length n.

Proof. Obviously every finite free resolution is stably free. Suppose now that M has a stably free resolution of length n:

$$0 \to E_n \to \cdots \to E_0 \to M \to 0$$

Choose any index $0 \leqslant i < i+1 \leqslant n$. There are finite free modules F_i, F_{i+1} corresponding to E_i, E_{i+1} respectively. Set $F = F_i \oplus F_{i+1}$. Then, we have a stably free resolution:

$$0 \to E_n \to \cdots \to E_{i+1} \oplus F \to E_i \oplus F \to E_{i-1} \to \cdots \to E_0 \to M \to 0,$$

with the modified map being $(E_{i+1} \to E_i, \mathbf{id}_F)$.

Applying the above construction successively to pairs (E_0, E_1), (E_1, E_2) and so on, we end up with a finite free resolution of M. ∎

Definition 1.5. M_1 and M_2 are said to be *stably isomorphic* if there exist finite free modules F_1 and F_2 such that $M_1 \oplus F_1 \cong M_2 \oplus F_2$.

Lemma 1.6 (Schanuel). Let $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ be exact sequences where P and P' are projective. Then $K \oplus P' \cong K' \oplus P$.

Proof. Treat K and K' as submodules of P and P' respectively. The projectivity of P and P' gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle w} & & \Vert{\scriptstyle \mathbf{id}} & & \\ 0 & \longrightarrow & K' & \longrightarrow & P' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where u is the restriction of w to K. Consider the sequence $0 \to K \xrightarrow{f} P \oplus K' \xrightarrow{g} P' \to 0$ where

$$f(x) = (x, u(x)) \quad \text{and} \quad g(y,z) = w(y) - z.$$

We contend that this is exact.

- Exactness at K is trivial.
- It is easy to see that $g \circ f = 0$. Suppose $(y,z) \in \ker g$, that is, $w(y) = z$. Since $z \in K'$, we must have that $y \in K$ whence $u(y) = z$, which proves exactness at $P \oplus K'$.
- Choose some $x' \in P'$. We can choose an $x \in P$ such that the images of x and x' in M are the same. Thus, $x' - w(x) \in K'$ whence exactness at P' follows.

Finally, since P' is projective, the sequence splits, giving us the desired conclusion. ∎

Lemma 1.7. Suppose M_1 and M_2 are stably isomorphic. Let

$$0 \to N_1 \to E_1 \to M_1 \to 0 \quad \text{and} \quad 0 \to N_2 \to E_2 \to M_2 \to 0$$

be exact sequences where E_1 and E_2 are stably free. Then N_1 is stably isomorphic to N_2.

Proof. There are finite free modules F_1, F_2 such that $M_1 \oplus F_1 \cong M_2 \oplus F_2$. We may modify the above short exact sequences to obtain

$$0 \to N_1 \to E_1 \oplus F_1 \to M_1 \oplus F_1 \to 0 \quad \text{and} \quad 0 \to N_2 \to E_2 \oplus F_2 \to M_2 \oplus F_2 \to 0.$$

Invoking Lemma 1.6,

$$N_1 \oplus E_2 \oplus F_2 \cong N_2 \oplus E_1 \oplus F_1.$$

Since both E_1, E_2 are stably free, there is a finite free module F such that both $E_1 \oplus F$ and $E_2 \oplus F$ are finite free. Thus,

$$N_1 \oplus (E_2 \oplus F \oplus F_2) \cong N_2 \oplus (E_1 \oplus F \oplus F_1)$$

and the conclusion follows. ■

DEFINITION 1.8. The minimal length of a stably free resolution of a module is called its *stably free dimension*.

THEOREM 1.9. Let M be a module admitting a stably free resolution

$$0 \to E_n \to \cdots \to E_0 \to M \to 0$$

of length n. Let

$$F_m \to \cdots \to F_0 \to M$$

be an exact sequence with F_i stably free for $0 \leqslant i \leqslant m$.

(a) If $m < n-1$, then there exists a stably free module F_{m+1} such that the above sequence can be continued exactly to

$$F_{m+1} \to F_m \to \cdots \to F_0 \to M$$

(b) If $m = n-1$ and $F_n = \ker(F_{n-1} \to F_{n-2})$. Then F_n is stably free.

Proof. For $0 \leqslant i \leqslant n$, define $K_i = \ker(E_i \to E_{i-1})$ with the convention that $E_{-1} = M$. Similarly, define $K_i' = \ker(F_i \to F_{i-1})$. Using Lemma 1.7, repeatedly along with the exact sequences

$$0 \to K_i \to E_i \to E_{i-1} \to 0 \quad \text{and} \quad 0 \to K_i' \to F_i \to K_{i-1}' \to 0,$$

we conclude that K_m and K_m' are stably isomorphic. Thus, there exist finite free modules F, F' such that $K_m \oplus F \cong K_m' \oplus F'$.

(a) $m < n-1$: We have

$$E_{m+1} \oplus F \twoheadrightarrow K_m \oplus F \cong K_m' \oplus F' \to K_m' \to 0.$$

Set $F_{m+1} = E_{m+1} \oplus F$ which is easily seen to be stably free.

(b) $m = n-1$: We can choose $K_m = E_n$. Then, $E_n \oplus F$ is stably free, whence so is $K_m' \oplus F'$, in particular, so is K_m'. This completes the proof. ■

Corollary. If $0 \to M_1 \to E \to M \to 0$ is exact, M has stably free dimension $\leqslant n$, and E is stably free, then M_1 has stably free dimension $\leqslant n-1$.

Proof. Let $0 \to E_n \to \cdots \to E_0 \to M \to 0$. We have an incomplete stably free resolution $E \to M \to 0$. We may now invoke Theorem 1.9 with $F_0 = E$ to obtain a resolution

$$0 \to F_n \to \cdots \to F_0 = E \to M \to 0.$$

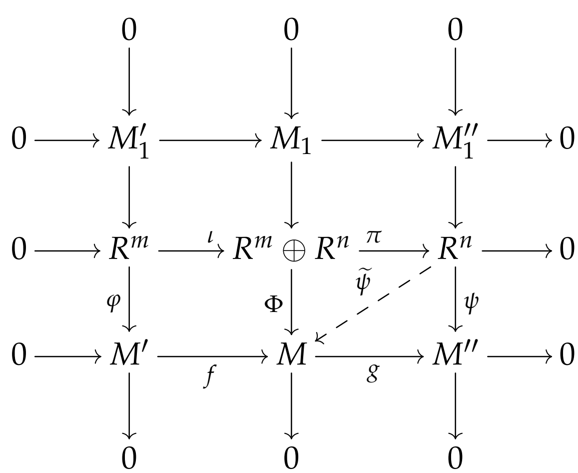
But note that $M_1 = \ker(E \to M)$ and hence, there is a stably free resolution

$$0 \to F_n \to \cdots \to F_1 \to M_1 \to 0,$$

and the conclusion follows. ■

Remark 1.10. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated modules. Then, there are surjections $\varphi : R^m \to M'$ and $\psi : R^n \to M''$, where R is the base ring. There is also the canonical injection $\iota : R^m \to R^m \oplus R^n$ and the canonical surjection $\pi : R^m \oplus R^n \to R^n$. Define the map $\Phi : R^m \oplus R^n \to M$ given by $\Phi(x,y) = f(\varphi(x)) + \widetilde{\psi}(y)$, where $\widetilde{\psi} : R^n \to M$ is a lift of the map $\psi : R^n \to M''$.

We contend that Φ is surjective. Indeed, let $m \in M$ then there is a $y \in R^n$ such that $\psi(y) = g(m)$. It is easy to see that $m - \widetilde{\psi}(y) \in \ker g = \operatorname{im} f$ and hence, there is an $x \in R^m$ such that $f \circ \varphi(x) = m - \widetilde{\psi}(y)$. It follows that $\Phi(x,y) = m$. Finally, the Snake Lemma gives a nice exact diagram.



Lemma 1.11. Let M'' be finitely presented and M finitely generated. If M' is the kernel of a surjection $M \twoheadrightarrow M''$, then M' is finitely generated.

Proof. We first prove this when M is finite free. Since M'' is finitely presented, there is an exact sequence $0 \to K \to F \to M'' \to 0$, where F is a finite free module and K is finitely generated. Due to Lemma 1.6, $M' \oplus F \cong K \oplus M$, whence M' is finitely generated.

Now, suppose M is just finitely generated. It can be written as the quotient of a free module $F \twoheadrightarrow M$. This gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & F & & & & \\
 & & \downarrow & & \downarrow & \searrow & & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

whence M' is finitely generated. ■

THEOREM 1.12. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence. If any two of these modules have a finite free resolution, then so does the third.

Proof. There are three possible cases. We shall tacitly use Proposition 1.4 throughout this proof.

M' **and** M**:** We induct on the stable free dimension of M. For the base case with the stable free dimension 0, M is stable free and the conclusion follows since M' too has a finite free resolution. Next, suppose the stable free dimension of M is $n \geqslant 1$. Due to Remark 1.10 and Corollary 1, the stable free dimension of M_1 is at most $n-1$ whence the induction hypothesis applies and M_1'' has a finite free resolution and the conclusion follows.

M' **and** M''**:** Induct on the maximum of the stable free dimension of M' and M''. The base case occurs when both M' and M'' have stably free dimension 0, that is, both are stably free, consequently, projective. It follows that $M \cong M' \oplus M''$ is stably free.

Next, for the induction step, using Remark 1.10 and Corollary 1 we see that the maximum of the stably free dimension of M_1' and M_1'' is at most $n-1$, whereby the induction hypothesis applies and the conclusion follows.

M **and** M''**:** We induct on the stably free dimension of M''. In the base case, M'' is stably free, in particular, projective, and hence, $M' \oplus M'' \cong M$, whence M' is also stably free.

As for the inductive step, again use Remark 1.10 and Corollary 1 to conclude. ■

§2 SERRE'S THEOREM

THEOREM 2.1. Let R be a Noetherian ring. If every finite R-module has a finite free resolution, then every finite $R[X]$-module has a finite free resolution.

Proof. Let M be a finite $R[X]$-module. There is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0,$$

where $M_i/M_{i+1} \cong R[X]/\mathfrak{P}_i$ for some prime $\mathfrak{P}_i$. In light of Theorem 1.12, it suffices to prove the theorem in the case $M = R[X]/\mathfrak{P}$ for some prime $\mathfrak{P}$.

Suppose the theorem is false. Let Σ be the collection of all primes $\mathfrak{P}$ such that $R[X]/\mathfrak{P}$ does not admit a finite free resolution. Choose $\mathfrak{P}$ in Σ that maximizes $\mathfrak{p} = \mathfrak{P} \cap R$.

Let $R_0 = R/\mathfrak{p}$, K_0 its quotient field, $\mathfrak{P}_0 = \mathfrak{P}/\mathfrak{p}R[X]$ and $M = R[X]/\mathfrak{P}$. We may view M as an $R_0[X]$-module, equal to $R_0[X]/\mathfrak{P}_0$. Let $f_1, \dots, f_n$ be a finite set of generators for $\mathfrak{P}_0$, and let f be a polynomial of minimal degree in $\mathfrak{P}_0$.

We c an write $f_i = q_i f + r_i$ for $1 \leqslant i \leqslant n$ with $q_i, r_i \in K_0[X]$ and $\deg r_i < \deg f$ or $r_i = 0$. Let d_0 be a common denominator for all coefficients of all q_i, r_i. Then, $d_0 \neq 0$ and

$$d_0 f_i = q_i' f + r_i',$$

where $q_i' = d_0 q_i, r_i' = d_0 r_i \in R_0[X]$ Since $\deg f$ is minimal in $\mathfrak{P}_0$, it follows that $r_i' = 0$ for all i, so $d_0 \mathfrak{P}_0 \subseteq (f)$.

Let $N_0 = \mathfrak{P}_0/(f)$, so that N_0 is a module over $R_0[X]$, and hence, N_0 can also be viewed as an $R[X]$-module. When so viewed, we denote N_0 by N. Let $d \in R$ be any element reducing to d_0 mod $\mathfrak{p}$. Since $d_0 \neq 0$, $d \notin \mathfrak{p}$.

The module N_0 has a filtration such that each successive quotient is isomorphic to $R_0[X]/\mathfrak{Q}_0$ where $\mathfrak{Q}_0$ is an associated prime of N_0. Let $\mathfrak{Q}$ be the pullback of $\mathfrak{Q}_0$ to $R[X]$. It is easy to argue that these prime ideals $\mathfrak{Q}$ are precisely the associated primes of N in $R[X]$. Since d_0 kills N_0, d must kill N and hence, d lies in every associated prime of N.

Note that each associated prime $\mathfrak{Q}$ of N contains $\mathfrak{P}$ and due to the preceding paragraph, $\mathfrak{Q} \cap R \supsetneq \mathfrak{P} \cap R$. Due to the maximality involved in the choice of $\mathfrak{P}$, every successive quotient in the filtration of N has a finite free resolution, whence N has a finite free resolution.

By assumption, $\mathfrak{p}$ has a finite free resolution as an R-module, say

$$0 \to E_n \to \cdots \to E_0 \to \mathfrak{p} \to 0.$$

Then

$$0 \to E_n[X] \to \cdots \to E_0[X] \to \mathfrak{p}[X] \to 0$$

is a finite free resolution of $\mathfrak{p}[X] \subseteq R[X]$ as an $R[X]$-module. From the exact sequence

$$0 \to \mathfrak{p}[X] \to R[X] \to R_0[X] \to 0,$$

it follows that $R_0[X]$ has a finite free resolution as an $R[X]$-module.

There is a surjective $R[X]$-linear map $\mu_f : R_0[X] \to (f)$, which is just multiplication by f. The kernel of this map is trivial since $R_0[X]$ is an integral domain. It follows that (f) too has a finite free resolution as an $R[X]$-module.

From the exact sequence of $R[X]$-modules

$$0 \to (f) \to \mathfrak{P}_0 \to N \to 0,$$

we conclude that $\mathfrak{P}_0$ has a finite free resolution as an $R[X]$-module. Next, from another exact sequence of $R[X]$-modules

$$0 \to \mathfrak{p}R[X] \to \mathfrak{P} \to \mathfrak{P}_0 \to 0,$$

it follows that $\mathfrak{P}$ has a finite free resolution as an $R[X]$-module. Finally from

$$0 \to \mathfrak{P} \to R[X] \to R[X]/\mathfrak{P} \to 0,$$

we conclude that $R[X]/\mathfrak{P}$ admits a finite free resolution as an $R[X]$-module, a contradiction. This completes the proof. ∎

THEOREM 2.2 (SERRE). Let k be a field. Every finite projective module over $k[X_1, \ldots, X_n]$ admits a finite free resolution. Equivalently, is stably free.

§3 UNIMODULAR POLYNOMIAL VECTORS

DEFINITION 3.1. Let A be a commutative ring. An n-tuple $(f_1, \ldots, f_n) \in A^n$ is said to be *unimodular* if they generate the unit ideal in A. A unimodular vector is said to have the *unimodular extension property* if there exists a matrix in $\mathrm{GL}_n(A)$ with $(f_1, \ldots, f_n)^\top$ as the first column.

REMARK 3.2. Note that a unimodular column vector $(f_1, \ldots, f_n)^\top$ has the unimodular extension property if and only if some column vector obtained after a series of row and column operations has that property.

THEOREM 3.3 (HORROCKS). Let $(\mathfrak{o}, \mathfrak{m}, k)$ be a local ring and $A = \mathfrak{o}[X]$. Let f be a unimodular column vector in $A^{(n)}$ such that some component in f has leading coefficient 1. Then f has the unimodular extension property.

Proof. If $n = 1$, then there is nothing to prove. Next, if $n = 2$, then $(f_1, f_2) = (1)$ and hence, there are $g_1, g_2 \in A$ such that $f_1 g_1 + f_2 g_2 = 1$, whence

$$\det \begin{pmatrix} f_1 & -g_2 \\ f_2 & g_1 \end{pmatrix} = 1.$$

Now, assume $n \geqslant 3$ and induct on the smallest degree d of a component of f with leading coefficient 1. The base case with $d = 0$ is trivial. Suppose now that $d \geqslant 1$. Using row operations, we may suppose that $\deg f_i < d$ for $i \neq 1$. Since there is a linear combination $\sum_{i=1}^{n} g_i f_i = 1$, not all coefficients of $f_2, \ldots, f_n$ can lie in $\mathfrak{m}$, for if they did, then $g_1 f_1 \equiv 1 \pmod{\mathfrak{m}}[X]$, which is absurd, since f_1 is not a unit modulo $\mathfrak{m}[X]$.

Without loss of generality, suppose that some coefficient of f_2 does not lie in $\mathfrak{m}$. Write

$$\begin{aligned} f_1(X) &= X^d + a_{d-1} X^{d-1} + \cdots + a_0 \quad a_i \in \mathfrak{o} \\ f_2(X) &= b_s X^s + \cdots + b_0 \quad b_i \in \mathfrak{o},\ s \leqslant d-1 \end{aligned}$$

such that some b_i is a unit. Lt $\mathfrak{a}$ be the ideal generated by all leading coefficients of polynomials of the form $g_1 f_1 + g_2 f_2$ of degree $\leqslant d-1$. We claim that $\mathfrak{a}$ contains all the b_i. This can be seen inductively. First, b_s lies in $\mathfrak{a}$ because of $X^{d-s} f_2(X)$. Next, b_{s-1} is realised as $X^{d-s} f_2(X) - b_s f_1(X)$ has leading coefficient $b_{s-1} - b_s a_{d-1}$. But since $\mathfrak{a}$ already contains b_s, it must also contain b_{s-1}. Continue this way. Recall that one of the b_i's is a unit and hence, $\mathfrak{a}$ is the unit ideal.

Thus, there is a linear combination $h = g_1 f_1 + g_2 f_2$ having degree $\leqslant d-1$ and leading coefficient 1. If $\deg f_3 < \deg h$, then $h + f_3$ has leading coefficient 1 and degree $\leqslant d-1$. Now suppose $\deg f_3 = \deg h$. If the leading coefficient of f_3 is a unit, then multiply by its inverse to make the leading coefficient 1. If, on the other hand, it is not a unit, then the leading coefficient of $h + f_3$ is a unit and hence, can be made 1 after multiplying by its inverse. Now, the induction hypothesis applies, thereby completing the proof. ■

DEFINITION 3.4. Let A be a commutative ring. For two column vectors $f, g \in A^{(n)}$, we write $f \sim g$ to mean that there exists $M \in \mathrm{GL}_n(A)$ such that $f = Mg$, and we say that f is *equivalent* to g over A.

PROPOSITION 3.5. Let $(\mathfrak{o}, \mathfrak{m}, k)$ be a local ring. Let f be a unimodular vector in $\mathfrak{o}[X]^{(n)}$ such that some component has leading coefficient 1. Then $f \sim f(0)$ over $\mathfrak{o}[X]$.

Proof. Note that $f(0) \in \mathfrak{o}^{(n)}$ has at least one component which is a unit, for if not, then the constant term of any linear combination would always lie in $\mathfrak{m}$. Hence, it follows that $f(0) \sim \mathbf{e}_1$. On the the other hand, due to Theorem 3.3, $f \sim \mathbf{e}_1$, thereby completing the proof. ■

LEMMA 3.6. Let R be an integral domain, and $S \subseteq R$ a multiplicatively closed subset containing 1. Let X and Y be independent variables. If $f(X) \sim f(0)$ over $S^{-1}R[X]$, then there is a $c \in S$ such that $f(X + cY) \sim f(X)$ over $R[X, Y]$.

Proof. Let $M \in \mathrm{GL}_n(S^{-1}R[X])$ be such that $f(X) = M(X)f(0)$. That is, $M(X)^{-1}f(X) = f(0)$. The right hand side is independent of X and hence, $M(X+Y)^{-1}f(X+Y) = f(0)$ when viewed over $S^{-1}R[X,Y]$. Set $G(X,Y) = M(X)M(X+Y)^{-1} \in S^{-1}R[X,Y]$, then $G(X,Y)f(X+Y) = f(X)$.

By construction, we have $G(X, 0) = I$, the identity matrix and hence, we can write $G(X,Y) = I + YH(X,Y)$ for some matrix $H(X,Y)$ with entries in $S^{-1}R[X,Y]$. There is some $c \in S$ such that cH has entries in $R[X,Y]$. Then, $G(X, cY)$ has entries in $R[X,Y]$. Now, since $\deg M(X)$ is invertible in $S^{-1}R[X]$, it must be an element of $S^{-1}R$. Further, since $\deg M(X + cY) = \det M(X)$, we have $\det G(X, cY) = 1$, thereby completing the proof. ■

THEOREM 3.7. Let R be an integral domain, and let f be a unimodular vector in $R[X]^{(n)}$, such that one component has leading coefficient 1. Then $f(X) \sim f(0)$ over $R[X]$.

Proof. Let J be the set of elements $c \in R$ such that $f(X + cY)$ is equivalent to $f(X)$ over $R[X,Y]$. We claim that J is an ideal.

- Let $c \in J$ and $a \in R$. Then, $f(X + caY) = f(X + c(aY))$ is equivalent to $f(X)$ over $R[X, aY]$, which is a subring of $R[X,Y]$, whence the equivalence holds over the latter too.

- Let $c, c' \in J$. Then $f(X + (c - c')Y)$ is equivalent to $f(X)$ over $R[X, (c-c')Y]$, which is again a subring of $R[X,Y]$, whence the equivalence holds over the latter too.

We next contend that J is the unit ideal. Suppose not, then we can choose a maximal ideal $\mathfrak{m}$ containing J. Due to Proposition 3.5, $f(X)$ is equivalent to $f(0)$ over $R_{\mathfrak{m}}[X]$, consequently, using Lemma 3.6, there is a $c \in R \setminus \mathfrak{m}$ such that $f(X+cY)$ is equivalent to $f(X)$ over $R[X,Y]$, a contradiction to the fact that $J \subseteq \mathfrak{m}$. Thus, J is the unit ideal in R and there exists an $M(X,Y) \in \mathrm{GL}_n(R[X,Y])$ such that $f(X+Y) = M(X,Y)f(X)$. Substituting $X=0$, we get $f(Y) = M(0,Y)f(X)$ where $M(0,Y)$ is also invertible and the conclusion follows. ∎

THEOREM 3.8. Let k be a field and f a unimodular vector in $k[X_1,\dots,X_n]^{(n)}$. Then f has the unimodular extension property.

Proof. The proof of this is quite similar to that of Noether Normalization. We induct on r. Suppose first that $r \geqslant 2$. Let $Y_r = X_r$ and $X_i = Y_i + Y_r^{N_i}$ for some suitable choice of N_i's such that at least one component of $g(Y_1,\dots,Y_r) = f(X_1,\dots,X_r)$ has leading coefficient equal to 1.

Due to Theorem 3.7, using the fact that $k[y_1,\dots,Y_{r-1}]$ is an integral domain, we have that

$$g(Y_1,\dots,Y_r) = M(Y_1,\dots,Y_r)g(Y_1,\dots,Y_{r-1},0)$$

where $M \in \mathrm{GL}_n\left(k[Y_1,\dots,Y_r]\right)$. Note that $g(Y_1,\dots,Y_{r-1},0)$ is unimodular over $k[Y_1,\dots,Y_{r-1}]$ and hence, has the unimodular extension property, whence so does $g(Y_1,\dots,Y_r)$. This completes the induction step.

Finally, we must handle the base case of $k[X]$, which is a PID. This is straightforward for if $f = (f_1,\dots,f_n)^\top$, then making repeated use of the Euclidean algorithm, we can make one of the components a unit, since $\gcd(f_1,\dots,f_n) = 1$. This completes the proof. ∎

DEFINITION 3.9. A (commutative) ring A is said to have the *unimodular extension property* if for every $n \geqslant 1$, every unimodular vector $f \in A^{(n)}$ has the property.the property.

LEMMA 3.10. Let A have the unimodular extension property. If E is a stably free A-module, then E is free.

Proof. Let F be a finite free module of rank m such that $E \oplus F$ is finite free. We first show that E is free when $m=1$. That is, $E \oplus A \cong A^{(n)}$ for some positive integer n. We may treat both E and A as submodules of $A^{(n)}$ and let $u^1 = (a_{11},\dots,a_{n1})^\top$ be a basis for the submodule A as an A-module. Consider the canonical projection $A^{(n)} \to A$. This map sends u^1 to 1 and being A-linear, it is of the form

$$(x_1,\dots,x_n)^\top \mapsto \alpha_1 x_1 + \cdots + \alpha_n x_n$$

for some $\alpha_1,\dots,\alpha_n \in A$. Thus, u^1 is unimodular.

The unimodular extension property furnishes an

$$M = (u^1,\dots,u^n) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathrm{GL}_n(A).$$

Using column operations, on M, one can make sure that $u^2, \dots, u^n \in E$ while maintaining $M \in \mathrm{GL}_n(A)$. Since $u^1, \dots, u^n$ form a basis for $A^{(n)}$, we see that $u^2, \dots, u^n$ must span E, whence E is free.

Finally, if F has rank $m \geqslant 2$, then write $F = F' \oplus A$ and use the first half of the proof to induct downwards. ■

THEOREM 3.11 (QUILLEN-SUSLIN). Let k be a field. Every finite projective module over $k[X_1, \dots, X_n]$ is free.

Proof. Let P be a projective module over $k[X_1, \dots, X_n]$. Due to Theorem 2.2, P is stably free. Next, due to Theorem 3.8 and Lemma 3.10, P must be free. ■