$\prod_{i=1}^{\infty} \mathbb{Z}$

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In $G = \prod_{i=1}^{\infty} \mathbb{Z}$, there are the "standard basis vectors" $\{e_i : i \geqslant 1\}$ given by

$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

These are linearly independent over $\mathbb{Z}$ and hence generate a free abelian subgroup of G, which we denote by H.

LEMMA 1. Let $f : G \to \mathbb{Z}$ be a homomorphism such that $f|_H = 0$. Then $f = 0$.

Proof. As a consequence of the hypothesis, it is easy to see that $f(a_1, a_2, \dots) = f(b_1, b_2, \dots)$ if $a_i = b_i$ for all but finitely many $i \geqslant 1$.

Let $p \geqslant 2$ be any prime and $(a_n)_{n \geqslant 1}$ be any sequence of integers. Our observation above yields that for all $n \geqslant 1$,

$$f(a_1 p, a_2 p^2, \dots) = f(\underbrace{0, \dots, 0}_{n-1 \text{ times}}, a_n p^n, a_{n+1} p^{n+1}, \dots).$$

The right hand side is divisible by p^n and hence, $p^n \mid f(a_1 p, a_2 p^2, \dots)$ for all $n \geqslant 1$. This is possible if and only if

$$f(a_1 p, a_2 p^2, \dots) = 0.$$

Finally, let $(a_n)_{n \geqslant 1} \in G$. Since 2^n and 3^n are coprime for all $n \geqslant 1$, there are $b_n, c_n \in \mathbb{Z}$ such that $a_n = b_n 2^n + c_n 3^n$. Thus,

$$f(a_1, a_2, \dots) = f(2b_1, 4b_2, \dots) + f(3c_1, 9c_2, \dots) = 0.$$

This completes the proof. ■

THEOREM 2. G is not a free abelian group.

Proof. Suppose G were free, then there is a set S and an isomorphism $\varphi : G \to F = \bigoplus_S \mathbb{Z}$. Since G is uncountable, so is S. Let $\pi_s : F \to \mathbb{Z}$ denote the projection onto the s-th coordinate.

For each $i \geqslant 1$, $\varphi(e_i)$ has only finitely many nonzero coordinates, say $S_i \subseteq S$. Then, $\bigcup_i S_i$ is a countable subset of S, whence $T = S \setminus \bigcup_i S_i$ is still uncountable, in particular, nonempty.

For each $t \in T$, $\pi_t \circ \varphi(e_i) = 0$ for all $i \geqslant 1$ and hence, $\pi_t \circ \varphi|_H = 0$. Due to the preceding lemma, $\pi_t \circ \varphi = 0$ for all $t \in T$, but this is absurd, since φ is surjective. ■

Lemma 3. If $f : G \to \mathbb{Z}$ is a homomorphism, then $f(e_i) = 0$ for all but finitely many $i \geqslant 1$.

Proof. Suppose not, then there is a sequence $1 \leqslant i_1 < i_2 < \cdots$ such that $f(e_{i_j}) \neq 0$. By composing f with a suitable endomorphism of G, we may suppose that $i_j = j$ and $f(e_{i_j}) > 0$ for all $j \geqslant 1$.

Let $a_i = f(e_i)$ and p be a prime not dividing a_1. Define two sequences $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ by setting $x_1 = 1$,

$$x_{n+1} = pf(x_1, \dots, x_n, 0, 0, \dots) \quad \text{for } n \geqslant 1,$$

and $y_1 = 1$,

$$y_n = f(x_1, \dots, x_{n-1}, 0, 0, \dots) \quad \text{for } n \geqslant 2.$$

Hence, $x_n = py_n$ for $n \geqslant 2$. Note that for $n \geqslant 2$

$$y_{n+1} = y_n + a_n x_n = y_n + pa_n y_n = y_n(1 + pa_n) \geqslant (p+1)y_n.$$

In particular, $y_n \to \infty$ as $n \to \infty$. Also, multiplying both sides of the above equation by p, we have

$$x_{n+1} = x_n(1 + pa_n) \quad \text{for } n \geqslant 2.$$

Therefore, for $n \geqslant 2$, $y_n \mid x_m$ whenever $m \geqslant n$. We can write

$$f(x_1, x_2, \dots) = y_n + f(0, \dots, 0, x_n, x_{n+1}, \dots).$$

Since y_n divides the right hand side, $y_n \mid f(x_1, x_2, \dots)$ for all $n \geqslant 1$. But y_n grows without bound, and hence, $f(x_1, x_2, \dots) = 0$. Finally, note that

$$0 = f(x_1, x_2, \dots) = a_1 + f(0, x_2, x_3, \dots) \equiv a_1 \pmod{p},$$

since $p \mid x_i$ for all $i \geqslant 2$. This is absurd, since $p \nmid a_1$. ■

Theorem 4. $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$ is a free abelian group with basis $\pi_i : G \to \mathbb{Z}$, the canonical projections.

Proof. Follows from Lemma 1 and Lemma 3 ■