Picard's Theorems

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Abstract

Taming the big bad wolves of $\mathbb{C}$omplex Analysis by nuking it.

§1 Analytic Covering Maps

Recall first the definiton of a covering map in a general topological space.

Definition 1.1 (Abstract Covering Map). A map $\pi : E \to B$ is said to be a *covering map* if there is an open cover $\{U_\alpha\}$ of B such that $\pi^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times D_\alpha$ where D_α is a topological space with the discrete topology.

Definition 1.2 (Analytic Covering Map). Let $\Omega, G \subseteq \mathbb{C}$ be open sets. An abstract covering map $\pi : \Omega \to G$ is said to be *analytic* if π is a holomorphic map.

Proposition 1.3. Let $\pi : \Omega \to G$ be an analytic covering map and $f : H \to G$ a holomorphic map. If there is a continuous map $\widetilde{f} : H \to \Omega$ such that $\pi \circ \widetilde{f} = f$, then $\widetilde{f}$ is holomorphic.

Proof. Let $z_0 \in H$. Then, there is a neighborhood U of $f(z_0)$ in G and a neighborhood V of $\widetilde{f}(z_0)$ such that π is a biholomorphism from V to U. Let W be a neighborhood of z_0 that maps into V under $\widetilde{f}$. Then, on W, we have $\widetilde{f} = \pi^{-1} \circ f$, which is holomorphic. ■

§2 Modular Function

Definition 2.1 (Modular Transformation). A *modular transformation* is a Möbius transformation

$$M(z) = \frac{az + b}{cz + d}$$

such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. The set of all modular transformations form a group, known as the *modular group*. We often identify this group with $\mathrm{SL}_2(\mathbb{Z})$.

DEFINITION 2.2. Let Γ denote the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by

$$\tau = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

This group will be of particular interest during the construction of a modular function. It is customary to denote this group by $\Gamma(2)$ but we drop the "(2)" for brevity.

DEFINITION 2.3. Let G denote the region

$$\{z = x + iy \in \mathbb{H} \colon -1 \leqslant x < 1, \ |2z-1| > 1 \text{ and } |2z+1| \geqslant 1\}.$$

THEOREM 2.4. Let G and Γ be as defined above. Then,

1. $\varphi_1(G) \cap \varphi_2(G) = \emptyset$ whenever $\varphi_1 \neq \varphi_2$ in Γ.
2. $\mathbb{H} = \bigsqcup_{\varphi \in \Gamma} \varphi(G)$.
3. $$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \ a, d \equiv 1 \pmod 2 \text{ and } b, c \equiv 0 \pmod 2 \right\}.$$

Proof. ■

THEOREM 2.5. Let G and Γ be as defined above. Then, there is a holomorphic function $\lambda : \mathbb{H} \to \mathbb{C}$ having the following properties:

1. $\lambda \circ \varphi = \lambda$ for all $\varphi \in \Gamma$.
2. λ is injective on G.
3. $\lambda(\mathbb{H}) = \mathbb{C} \backslash \{0, 1\}$.
4. $\lambda : \mathbb{H} \to \mathbb{C} \backslash \{0, 1\}$ is a covering map.

Proof. Let

$$G_0 = \{z = x + iy \in \mathbb{H} \colon 0 < x < 1 \text{ and } |2z-1| > 1\}.$$

Note that G_0 is simply connected and thus, there is a conformal equivalence $f_0 : G_0 \to \mathbb{H}$. Then, there is an extension of f to a homeomorphism $f : \overline{G}_0 \to \overline{\mathbb{H}}$ that maps $\partial G_0 \to \partial \mathbb{H}$. Upon composing with a suitable Möbius transformation, we may suppose that $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Consider the following three pieces of ∂G_0,

$$\begin{aligned} L_1 &= \{z \in \overline{\mathbb{H}} \colon \Re(z) = 0\} \\ L_2 &= \{z \in \overline{\mathbb{H}} \colon |2z-1| = 1\} \\ L_3 &= \{z \in \overline{\mathbb{H}} \colon \Re z = 1\}. \end{aligned}$$

First, note that f is a bijection $L_1 \cup L_2 \cup L_3 \to \partial \mathbb{H} = \mathbb{R}$. Further, L_1 and L_3 must map to half lines with 0 and 1 mapping to themselves. Therefore, L_1 must map to $(-\infty, 1]$, L_2 to

$[0,1]$ and L_3 to $[1,\infty)$. Next, note that $f(L_1) \subseteq \mathbb{R}$ and hence, due to the Schwarz Reflection Principle, there is an extension of f to all of G, defined by $f(-x+iy) = \overline{f(x+iy)}$. Note that this gives $f(G) = \mathbb{C}\backslash\{0,1\}$ and $f(\operatorname{int} G) = \mathbb{C}\backslash[0,\infty)$. Finally, define $\lambda : \mathbb{H} \to \mathbb{C}$ by

$$\lambda(z) = \lambda(\varphi^{-1}(z)) \text{ when } z \in \varphi(G).$$

We contend that the λ defined above is holomorphic. Consider the set $\Delta = G \cup \sigma^{-1}(G) \cup \tau^{-1}(G)$, whose interior contains G. It is not hard to argue, from the definition of λ that it is continuous on Δ and holomorphic on the interiors of the aforementioned three sets that form it. Therefore, λ is holomorphic on the interior of Δ, in particular, on G.

Lastly, we show that λ is a covering map. To do this, we shall show that every point in $\mathbb{C}\backslash\{0,1\}$ has an evenly covered neighborhood. First, suppose $\zeta \in \mathbb{C}\backslash[0,\infty)$ and choose $\delta > 0$ small enough so that $B_0 = B(\zeta,\delta) \subseteq \mathbb{C}\backslash[0,\infty)$ and $U = f^{-1}(B_0) \subseteq G$. Obviously, $\lambda^{-1}(B_0) = \bigsqcup_{\varphi\in\Gamma} \varphi(U)$. Thus B_0 is an evenly covered neighborhood of ζ.

Next, suppose $t \in (0,1)$ and choose $\delta > 0$ small enough so that $B_0 = B(t,\delta) \subseteq \mathbb{C}\backslash\{0,1\}$. From the explicit definition of f, note that $f^{-1}(t)$ contains two points, $\{z_+, z_-\}$ and $f^{-1}(B_0)$ contains two components U_+ and U_- containing z_+ and z_- respectively, such that $f^{-1}(B \cap \pm\overline{\mathbb{H}}) = U_\pm$. The transformation σ defined previously maps $|2z+1| = 1$ to $|2z-1| = 1$, z_- to z_+ and hence, maps U_- to U_+. Consequently, $U_0 = U_+ \cup \sigma(U_-)$ is a neighborhood of z_+ such that $\lambda(U_0) = \lambda(U_+) \cup \lambda(\sigma(U_-)) = B_0$. Consequently, the components of $\lambda^{-1}(B_0)$ that are biholomorphically mapped to B_0 are $\varphi(U_0)$ where $\varphi \in \Gamma$.

Finally, suppose $t \in (1,\infty)$. Recall that L_3 is mapped to $[1,\infty)$ under f, which was initially defined on $\overline{G}_0$. Arguing as in the previous paragraph, we see that there are two points $z_\pm$ with neighborhoods $U_\pm$ that are mapped to one another under τ. Thus, it follow again, that t has an evenly covered neighborhood. This completes the proof. ■

COROLLARY. There is a covering map $\mu : \mathbb{D} \to \mathbb{C}\backslash\{0,1\}$

§3 NORMAL FAMILIES

THEOREM 3.1 (MONTEL-CARATHÉODORY). Let $\Omega \subseteq \mathbb{C}$ be a region and

$$\mathscr{F} = \{f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic and } f(\Omega) \subseteq \mathbb{C}\backslash\{0,1\}\}.$$

Then, $\mathscr{F}$ is a normal family in $C(\Omega, \widehat{\mathbb{C}})$.

Proof. To prove that $\mathscr{F}$ is normal, it suffices to show that for every disk D in Ω, the restriction of $\mathscr{F}$ to D is normal. Hence, we may suppose without loss of generality that $\Omega = \mathbb{D}$. To show normality, we shall show that every sequence of functions in $\mathscr{F}$ has a subsequence that is uniformly bounded on compact subsets of $\mathbb{D}$ or has a subsequence that converges uniformly to ∞ on compact subsets of $\mathbb{D}$.

Let $\{f_n\}$ be a sequence of functions in $\mathscr{F}$. Then, there is a point $\alpha \in \widehat{C}$ such that a subsequence $\{f_{n_k}\}$ of $\{f_n(0)\}$ converges to α. Replace $\{f_n\}$ by $\{f_{n_k}\}$.

Case 1: $\alpha \in \mathbb{C}\backslash\{0,1\}$.

Consider the analytic covering map $\mu : \mathbb{D} \to \mathbb{C}\backslash\{0,1\}$ and let U be an evenly covered neighborhood of α. Pick a component V of $\mu^{-1}(U)$. Since $\mathbb{D}$ is simply connected, there are holomorphic lifts $\widetilde{f}_n : \mathbb{D} \to \mathbb{D}$ such that $\mu \circ \widetilde{f}_n = f_n$ and $\widetilde{f}_n(0) \in V$ for sufficiently large n. This is a sequence of functions that is uniformly bounded on compact subsets of $\mathbb{D}$ and hence, has a subsequence $\{\widetilde{f}_{n_k}\}$ that converges to a holomorphic function $f : \mathbb{D} \to \mathbb{C}$. Note that $|f(z)| \leqslant 1$ for all $z \in \mathbb{D}$ and if $|f(z)| = 1$ for some $z \in \mathbb{D}$, then f must be a constant function β due to the Maximum Modulus Principle. In particular, this means that $\widetilde{f}_{n_k}(0) \to \beta$. Note that $\mu|_V$ is a biholomorphism and hence, admits a holomorphic (in particular, continuous) inverse. Then, we have

$$(\mu|_V)^{-1}(\alpha) = \lim_{k\to\infty} (\mu|_V)^{-1}(f_{n_k}(0)) = \beta,$$

which is absurd, since $\beta \notin \mathbb{D}$. Hence, $|f(z)| < 1$ for all $z \in \mathbb{D}$.

We shall now show that $\{f_{n_k}\}$ is uniformly bounded on compact subsets of $\mathbb{D}$, whence we would be done by Montel's Theorem. Let $K \subseteq \mathbb{D}$ be a compact set. Then, there is an $M < 1$ such that $|f(z)| \leqslant M$ on K. Choose $M < r < 1$. Then, for sufficiently large k, we have $|f(z) - \widetilde{f}_{n_k}(z)| < r - M$. Thus, for all such k, we have $|\widetilde{f}_{n_k}(z)| < r$. Note that μ is bounded on $\overline{B}(0, r)$ and hence, $f_{n_k} = \mu \circ \widetilde{f}_{n_k}$ is uniformly bounded on K.

Case 2: $\alpha = 1$.

Since $\mathbb{D}$ is simply connected and f_n never vanishes on $\mathbb{D}$ for all n, there is a "square root" $g_n : \mathbb{D} \to \mathbb{C}$. Replacing g_n by $-g_n$ if necessary, we may suppose that $g_n(0) \to -1$ as $n \to \infty$. Further, note that the g_n's have image contained in $\mathbb{C}\backslash\{0, 1\}$. From our analysis in **Case 1**, there is a subsequence $\{g_{n_k}\}$ that converges uniformly on compact subsets of $\mathbb{D}$. Since $f_{n_k} = g_{n_k}^2$, we are done by once again invoking Montel's Theorem.

Case 3: $\alpha = 0$. Simply replace f_n by $1 - f_n$. This brings us to **Case 2**.

Case 4: $\alpha = \infty$.

Let $g_n = 1/f_n$, which are holomorphic on $\mathbb{D}$ since f_n's never vanish on $\mathbb{D}$ for all n. Since the images of the g_n's are contained in $\mathbb{C}\backslash\{0, 1\}$, invoking the analysis of the previous cases, there must be a subsequence $\{g_{n_k}\}$ that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function $g : \mathbb{D} \to \mathbb{C}$. Note that $g(0) = 0$ but the g_n's have no zeros and hence, due to Hurwitz's Theorem, g must identically be 0. It follows that $f_{n_k}(z) \to \infty$ uniformly on compact subsets of $\mathbb{D}$. ■

§4 PICARD'S THEOREMS

THEOREM 4.1 (LITTLE PICARD). Let f be an entire function. If there are two distinct complex numbers that are not in the image of f, then f must be constant.

Proof. Without loss of generality, suppose f misses 0 and 1. Recall the analytic covering map $\mu : \mathbb{D} \to \mathbb{C}\backslash\{0, 1\}$. There is a holomorphic lift $\widetilde{f} : \mathbb{C} \to \mathbb{D}$ of f. Due to Liouville, $\widetilde{f}$ must be constant and hence, so must f. ■

THEOREM 4.2 (GREAT PICARD). Let $f : \Omega \to \mathbb{C}$ have an essential singularity at $0 \in \Omega$. Then, there is an $\alpha \in \mathbb{C}$ such that for all $\zeta \neq \alpha$, the equation $f(z) = \zeta$ has infinitely many solutions in any punctured neighborhood of 0 that is contained in Ω.

Proof. Suppose there is an $R > 0$ such that $B(0, R) \subseteq \Omega$ and $f(B(0, R))$ misses atleast two points in $\mathbb{C}$. We may suppose without loss of generality that 0 and 1 are missed. Note that we may also choose $R < 1/2$.

Since 0 is not a pole, the limit $|f(z)|$ as $z \to 0$ does not tend to ∞. Consequently, there is a positive constant $P > 0$ such that for all $R > \delta > 0$, there is a $z \in B(0, \delta)$ with $|f(z)| \leqslant P$. Begin with $\delta = R$ and choose such a z_1. Next, set $\delta = z_1$ and pick a corresponding z_2 and continue in this fashion. The sequence $\{z_i\}$ is bounded and hence, has a convergent subsequence, say $\{z_{n_k}\}$. Call this sequence $\{\alpha_k\}$.

Define $f_n : \Omega \to \mathbb{C}$ by $f_n(z) = f(2\alpha_n z/R)$. Then, due to Theorem 3.1 $\{f_n\}$ is a normal family and hence, admits a subsequence $\{f_{n_k}\}$ that either converges uniformly on compact subsets of Ω to either a holomorphic function $g : \Omega \to \mathbb{C}$ or to the identically ∞ function on Ω.

Suppose the former case and let $M = \max\{|g(z)| \colon |z| = R/2\}$. Due to uniform convergence on compact subsets of Ω, there is a k_0 such that for all $k \geqslant k_0$, we have $|f_{n_k}(z) - g(z)| \leqslant M$ whenever $|z| = R/2$ and hence, $|f(\alpha_{n_k} z)| = |f_{n_k}(z)| \leqslant 2M$ whenever $|z| = R/2$. Due to the Maximum Modulus Principle, $f(z)$ is bounded by $2M$ on the annulus $|\alpha_{n_k}| < |z| < R/2$. Since $|\alpha_{n_k}|$ grows arbitrarily small, we see that $f(z)$ Is bounded by $2M$ on the annulus $0 < |z| < R/2$. This would mean that $z = 0$ is a removable singularity, a contradiction.

Consider the latter case, $g \equiv \infty$. But this is obviously not possible since for sufficiently large n, $f_n(R/2)$ converges to a finite limit. This completes the proof. ■