The Universal Enveloping Algebra and The Poincaré-Birkhoff-Witt Theorem

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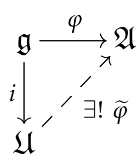
Abstract

In this article, we define and construct the universal enveloping algebra of a Lie algebra, then, state and prove the Poincaré-Birkhoff-Witt Theorem.

§1 The Universal Enveloping Algebra

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra over k. A *universal enveloping algebra* is a pair $(\mathfrak{U}, i)$ where $\mathfrak{U}$ is an associative algebra (over k, with identity) and $i : \mathfrak{g} \to \mathfrak{U}$ is a homomorphism of Lie algebras such that for any associative algebra $\mathfrak{A}$ (over k, with identity) and any Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{A}$, there is a unique k-algebra homomorphism $\widetilde{\varphi} : \mathfrak{U} \to \mathfrak{A}$ making the following diagram commute.



§§ Construction

Let $\mathfrak{T}$ denote the *tensor algebra* over $\mathfrak{g}$, that is,

$$\mathfrak{T} = \bigoplus_{n \geqslant 0} \mathfrak{g}^{\otimes n}.$$

There is a map $\mu : \mathfrak{g}^{\otimes n} \times \mathfrak{g}^{\otimes m} \to \mathfrak{g}^{\otimes m+n}$ given by

$$\mu(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \ldots x_n \otimes y_1 \otimes \cdots \otimes y_m$$

and extending linearly. This gives $\mathfrak{T}$ the structure of a k-algebra.

Let $\mathfrak{K}$ denote the ideal in $\mathfrak{T}$ generated by all elements of the form

$$[x, y] - x \otimes y + y \otimes x$$

for $x, y \in \mathfrak{g}$. Set $\mathfrak{U} = \mathfrak{T}/\mathfrak{K}$ and let $\iota : \mathfrak{g} \to \mathfrak{U}$ be the composition

$$\mathfrak{g} \longrightarrow \mathfrak{T} \xrightarrow{\pi} \mathfrak{U}.$$

THEOREM 1.2. $(\mathfrak{U}, \iota)$ is a universal enveloping algebra for $\mathfrak{g}$.

Proof. Let $\varphi : \mathfrak{g} \to \mathfrak{A}$ be a homomorphism of Lie algebras where $\mathfrak{A}$ is an associative k-algebra. The universal property of the tensor algebra extends this to a k-algebra homomorphism $\widetilde{\varphi} : \mathfrak{T} \to \mathfrak{A}$.

Note that

$$\widetilde{\varphi}(x \otimes y - y \otimes x) = \widetilde{\varphi}(x \otimes y) - \widetilde{\varphi}(y \otimes x) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x) = \varphi([x, y]),$$

whence $\widetilde{\varphi}$ vanishes on $\mathfrak{K}$, thereby inducing a unique (due to the universal property of the kernel) map $\widetilde{\widetilde{\varphi}} : \mathfrak{U} \to \mathfrak{A}$, thereby completing the proof. ■

§2 THE POINCARÉ-BIRKHOFF-WITT THEOREM

Let $u_1, \ldots, u_n$ be a k-basis of $\mathfrak{g}$. A *monomial* in $\mathfrak{T}$ is an element of the form

$$u_{i_1} \otimes \cdots \otimes u_{i_n}$$

for $n \geqslant 1$. The number n is said to be the *degree* of the monomial. The *index* of the monomial is given by

$$\operatorname{ind}(u_{i_1} \otimes \cdots \otimes u_{i_n}) = \sum_{j<k} \eta_{jk}$$

where

$$\eta_{jk} = \begin{cases} 0 & i_j \leqslant i_k \\ 1 & i_j > i_k \end{cases}$$

A monomial is said to be *standard* if its index is 0. Let $\mathfrak{g}_n$ denote the vector space spanned by monomials of degree n and let $\mathfrak{g}_{n,i}$ denote the subspace of $\mathfrak{g}_n$ spanned by monomials of degree n and index $\leqslant i$.

LEMMA 2.1. Every element of $\mathfrak{T}$ is congruent modulo $\mathfrak{K}$ to a k-linear combination of 1 and standard monomials.

Proof. Straightforward induction on the index and degree of standard monomials. ■

Let $\mathfrak{P}$ denote the vector space spanned by $u_{i_1} \ldots u_{i_n}$ where $i_1 \leqslant \ldots \leqslant i_n$. These are to be interpreted as formal symbols without meaning.

LEMMA 2.2. There is a k-linear map $\sigma : \mathfrak{T} \to \mathfrak{P}$ such that

$$\sigma(1) = 1 \quad \text{and} \quad \sigma(u_{i_1} \otimes \cdots \otimes u_{i_n}) = u_{i_1} \ldots u_{i_n}.$$

if $i_1 \leqslant \ldots \leqslant i_n$. Further,

$$\sigma(u_{j_1} \otimes \cdots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \cdots \otimes u_{j_n}) = \sigma(u_{j_1} \otimes \cdots \otimes u_{j_n} - u_{j_1} \otimes \ldots u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots u_{j_n}).$$

Proof. We induct on degree and index, in that order. Suppose a linear map σ has been defined on $k \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$. It is easy to extend this to $k \oplus \cdots \oplus \mathfrak{g}_{n,0}$ by setting

$$\sigma(u_{i_1} \otimes \dots u_{i_n}) = u_{i_1} \dots u_{i_n}.$$

Now, suppose σ has already been defined for $k \oplus \dots \mathfrak{g}_{n,i-1}$. Suppose $j_k > j_{k+1}$. Then, define

$$\sigma(u_{j_1} \otimes \cdots \otimes u_{j_n}) = \sigma(u_{j_1} \otimes \cdots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \cdots \otimes u_{j_n}) + \sigma(u_{j_1} \otimes \cdots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \cdots \otimes u_{j_n}).$$

The right hand side is well-defined because the first term on the right has index at most $i-1$ and the second term on the right is a linear combination of monomials of smaller degree.

We must show that this is a well-defined assignment of σ, that is the right hand choice is independent of the pair of inversion chosen. To this end, let $j_l > j_{l+1}$. We must consider two cases.

Case 1: $l > k+1$. Set $u_{j_k} = u$, $u_{j_{k+1}} = v$, $u_{j_l} = w$, $u_{j_{l+1}} = x$.

We would like to show

$$\begin{aligned} &\sigma(\dots v \otimes u \otimes \cdots \otimes w \otimes x \dots) + \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes w \otimes x \dots) \\ &= \\ &\sigma(\dots u \otimes v \otimes \dots x \otimes w \dots) + \sigma(\dots u \otimes v \otimes \cdots \otimes [x,w] \otimes \dots). \end{aligned}$$

We can expand the left hand side of the above equality using the induction hypothesis as

$$\begin{aligned} &\sigma(\dots v \otimes u \otimes \cdots \otimes x \otimes w \dots) + \sigma(v \otimes u \otimes \cdots \otimes [x,w] \otimes \dots) \\ &+ \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes x \otimes w \dots) + \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes [w,x] \otimes \dots). \end{aligned}$$

The right hand side can be written as

$$\begin{aligned} &\sigma(\dots v \otimes u \otimes \cdots \otimes x \otimes w \dots) + \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes x \otimes w \dots) \\ &+ \sigma(\dots v \otimes u \otimes \cdots \otimes [x,w] \otimes \dots) + \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes [x,w] \otimes \dots). \end{aligned}$$

This completes the proof in this case.

Case 2: $l = k+1$. We write $u_{j_k} = u$, $u_{j_{k+1}} = v = u_{j_l}$ and $u_{j_{l+1}} = w$. We want to show the equality

$$\sigma(\dots v \otimes u \otimes w \dots) + \sigma(\dots [u,v] \otimes w \dots) = \sigma(\dots u \otimes w \otimes v \dots) + \sigma(\dots u \otimes [v,w] \dots).$$

The left hand side can be expanded further as

$$\begin{aligned} &\sigma(\dots v \otimes w \otimes u \dots) + \sigma(\dots v \otimes [u,w] \dots) + \sigma(\dots [u,v] \otimes w \dots) \\ &= \sigma(\dots w \otimes v \otimes u \dots) + \sigma(\dots [v,w] \otimes u \dots) + \sigma(\dots v \otimes [u,w] \dots) + \sigma(\dots [u,v] \otimes w \dots). \end{aligned}$$

Similarly, the right hand side can be expanded as

$$\begin{aligned}&\sigma(\dots w\otimes u\otimes v\dots)+\sigma(\dots[u,w]\otimes v\dots)+\sigma(\dots u\otimes[v,w]\dots)\\&=\sigma(\dots w\otimes v\otimes u\dots)+\sigma(\dots w\otimes[u,w]\dots)+\sigma(\dots[u,w]\otimes v\dots)+\sigma(u\otimes[v,w]\dots).\end{aligned}$$

It remains to show the equality:

$$\begin{aligned}&\sigma(\dots[v,w]\otimes u\dots)+\sigma(\dots v\otimes[u,w]\dots)+\sigma(\dots[u,v]\otimes w\dots)\\&=\sigma(\dots w\otimes[u,w]\dots)+\sigma(\dots[u,w]\otimes v\dots)+\sigma(u\otimes[v,w]\dots),\end{aligned}$$

which reduces to

$$\sigma(\dots[[v,w],u]\dots)+\sigma(\dots[v,[u,w]]\dots)+\sigma(\dots[[u,v],w]\dots)=0,$$

which follows from Jacobi's Identity. This completes the proof in this case.

Now that σ is well-defined for monomials, we can extend it linearly to $\mathfrak{g}_{n,i}$, thereby completing the induction. ∎

THEOREM 2.3 (POINCARÉ-BIRKHOFF-WITT THEOREM). The cosets of 1 and the standard monomials form a basis for $\mathfrak{U}=\mathfrak{T}/\mathfrak{K}$.

Proof. We have shown that the standard monomials and 1 span $\mathfrak{U}$. It remains to show linear independence. This follows from the preceding lemma, since the $u_{i_1}\dots u_{i_n}$'s are linearly independent in $\mathfrak{P}$. ∎

§3 PROPERTIES OF THE UNIVERSAL ENVELOPING ALGEBRA

DEFINITION 3.1. A ring R is said to be *filtered* if it is equipped with an increasing sequence $\mathscr{R}=\{R_i\}_{i\geqslant 0}$ of abelian subgroups such that

(a) $\bigcup_{i\geqslant 0} R_i = R$.

(b) For all $i,j\geqslant 0$, $R_iR_j\subseteq R_{i+j}$.

Each filtration of a ring gives rise to an *associated graded ring*,

$$\mathrm{Gr}_{\mathscr{R}}(R)=\bigoplus_{i\geqslant 0} R_i/R_{i-1},$$

with the convention that $R_{i-1}=0$.

For any $a\in R$, there is a non-negative integer n such that $a\in R_n$ but $a\notin R_{n-1}$. The homogeneous element $\overline{b}=b+R_{n-1}\in\mathrm{Gr}_{\mathscr{R}}(R)$ is called the *leading term* of b. If $b=0$, we take its leading term to be 0.

LEMMA 3.2. Let R be a filtered ring with an increasing filtration $\{R_i\}_{i\geqslant 0}$ and let G denote the corresponding associated graded.

(a) If G is a domain, then so is R.

(b) If G is left (resp. right) noetherian, then so is R.

Proof. (a) Suppose $a, b \in R \setminus \{0\}$ such that $ab = 0$ in R. Let $\overline{a}, \overline{b}$ denote the leading terms of a and b respectively. Then, $\overline{a}\overline{b} = 0$, a contradiction.

(b) Let I be a left-ideal in R. We shall show that I is finitely generated. Let $\overline{I}$ denote the abelian group generated by the leading terms of elements of I. It is easy to see that $\overline{I}$ is a left ideal in G (whence, is a homogeneous left ideal). Since G is left noetherian, there are $b_1, \ldots, b_n \in I$ such that $\overline{b}_1, \ldots, \overline{b}_n$ generate $\overline{I}$ as a left ideal in G where $\overline{b}_i$ is the leading term of b_i.

We contend that the b_i's generate I. Let $b \in I$. Then, $\overline{b}$, the leading term of b is a linear combination of the form

$$\overline{b} = \sum_i \overline{a}_i \overline{b}_i$$

where $\overline{a}_i \in G$. Since the left hand side is homogeneous, we may choose the a_i's to be homogeneous in G, consequently, the $\overline{a}_i$'s are leading terms of some $a_i \in R$.

From the above equality, we deduce that $\overline{b}$ is the leading term of $\sum_i a_i b_i$ whence, $b - \sum_i a_i b_i$ has leading term of homogeneous degree smaller than that of $\overline{b}$. An induction argument finishes the proof. ∎

If $\mathfrak{g}$ is a finite-dimensional Lie algebra over k (no restriction), then its universal enveloping algebra $\mathfrak{U}$ is equipped with a canonical filtration:

$$\mathfrak{U}^{(n)} = k \oplus \mathfrak{g} \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^n.$$

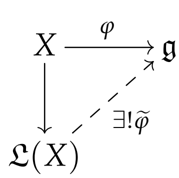
Using the Poincaré-Birkhoff-Witt theorem, it is not hard to see that the associated graded corresponding to the above filtration is isomorphic to $k[X_1, \ldots, X_n]$ where $n = \dim_k \mathfrak{g}$.

THEOREM 3.3. The universal enveloping algebra of a finite-dimensional Lie algebra over k is a left (and right) Noetherian domain.

Proof. Follows from Lemma 3.2 and the discussion above. ∎

§4 FREE LIE ALGEBRAS

DEFINITION 4.1. Let X be a set. A *free algebra over k* on X is a pair $(\mathfrak{L}(X), \iota)$ where $\iota : X \to \mathfrak{L}(X)$ is such that for any map of sets $\varphi : X \to \mathfrak{g}$ where $\mathfrak{g}$ is a Lie algebra over k, there is a unique Lie algebra homomorphism $\widetilde{\varphi} : \mathfrak{L}(X) \to \mathfrak{g}$ satisfying



§§ Construction

Let $\mathfrak{F}(X)$ denote the free k-algebra generated by X. Then, $\mathfrak{F}(X)$ has the structure of a Lie algebra. Let $\mathfrak{L}(X)$ denote the Lie subalgebra of $\mathfrak{F}(X)$ generated by X. We contend that $X \hookrightarrow \mathfrak{L}(X)$ is the free algebra on X.

Let $\varphi : X \to \mathfrak{g}$ be a map of sets where $\mathfrak{g}$ is a Lie algebra over k. Then, we have the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathfrak{g} \\ \downarrow & & \downarrow \\ \mathfrak{F}(X) & \underset{\exists! \widetilde{\varphi}}{\dashrightarrow} & \mathfrak{U}(\mathfrak{g}) \end{array}$$

Where $\widetilde{\varphi}$ restricts to φ on $X \subseteq \mathfrak{F}(X)$. Note that φ is also a Lie algebra homomorphism. Therefore, $\widetilde{\varphi}^{-1}\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{F}(X)$ containing X and hence, $\mathfrak{L}(X) \subseteq \widetilde{\varphi}^{-1}\mathfrak{g}$. It follows that $\widetilde{\varphi}$ restricts to a Lie algebra homomorphism $\widetilde{\varphi} : \mathfrak{L}(X) \to \mathfrak{g}$. The uniqueness follows since X generates $\mathfrak{L}(X)$. This completes the proof of existence.

The above discussion also shows:

PROPOSITION 4.2. $\mathfrak{F}(X)$ is the universal enveloping algebra of $\mathfrak{L}(X)$.

§5 EPIMORPHISMS OF LIE ALGEBRAS

DEFINITION 5.1. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras with a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$. Then, there is a Lie algebra structure on $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g}$ given by

$$[(h,g),(h',g')] = \Big([h,h'] + \psi_g(h') - \psi_{g'}(h), [g,g']\Big).$$

This is the *semidirect product* and is often denoted by $\mathfrak{h} \rtimes_\psi \mathfrak{g}$

LEMMA 5.2. Suppose $\mathfrak{h} \subsetneq \mathfrak{g}$ is an inclusion of Lie algebras and V a $\mathfrak{g}$-module with a $0 \neq \psi \in V$ that is annihilated by $\mathfrak{h}$ but not by $\mathfrak{g}$. Then the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is not an epimorphism.

Proof. We can treat V as an abelian Lie algebra and consider the semidirect product $\mathfrak{k} = V \rtimes \mathfrak{g}$. Note that since V is abeilan, every k-linear map $V \to V$ is a derivation of V.

Define the map $\theta : \mathfrak{g} \to V \rtimes \mathfrak{g}$ by $\theta(x) = (x \cdot \psi, x)$. We contend that this is a Lie algebra homomorphism. Indeed, for $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}[\theta(x), \theta(y)] &= [(x \cdot \psi, x), (y \cdot \psi, y)] \\ &= (x \cdot (y \cdot \psi) - y \cdot (x \cdot \psi), [x,y]) \\ &= [x,y] \cdot \psi.\end{aligned}$$

Further, for $x \in \mathfrak{h}$, $\theta(\mathfrak{h}) = (0, x)$. Consequently, the two maps $\theta : \mathfrak{g} \to V \rtimes \mathfrak{g}$ and $\iota : \mathfrak{g} \hookrightarrow V \rtimes \mathfrak{g}$ agree on $\mathfrak{h}$ but not on $\mathfrak{g}$. Thus, the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is not an epimorphism. ∎

THEOREM 5.3. Epimorphisms in the category of Lie algebras (including the infinite-dimensional ones) are precisely the surjective Lie algebra homomorphisms.

Proof. Obviously, surjecive Lie algebra homomorphisms $\mathfrak{h} \to \mathfrak{g}$ are epimorphisms. Therefore, it suffices to show that a proper inclusion $\mathfrak{h} \subsetneq \mathfrak{g}$ of Lie algebras is not epimorphic. We prove this in the case $\mathfrak{g}$ is finite-dimensional but an analogous proof works in the infinite-dimensional case.

Choose a k-basis $x_1, \dots, x_n$ of $\mathfrak{g}$ such that $x_{p+1}, \dots, x_n$ is a k-basis of $\mathfrak{h}$. Let $\mathfrak{U}$ denote the universal enveloping algebra of $\mathfrak{g}$. Recall that the Poincaré-Birkhoff-Witt theorem guarantees a k-basis of $\mathfrak{U}$ in the form $x_1^{m_1} \cdots x_n^{m_n}$ where $m_i \geqslant 0$ for $1 \leqslant i \leqslant n$. We use the notation $\mathbf{x}^m$ to denote products of the aforementioned kind.

Let V be the subspace of $\mathfrak{U}$ spanned by $\mathbf{x}^k$ where $k_{p+1} = \cdots = k_n = 0$, and let $\pi : \mathfrak{U} \twoheadrightarrow V$ denote the projection. Define a $\mathfrak{g}$-action on V as follows: For $v \in V \subseteq \mathfrak{U}$ and $x \in \mathfrak{g}$, set $x \cdot v = \pi(xv)$, where xv is the standard product in the associative algebra $\mathfrak{U}$.

First, we must show that this is gives V the structure of a $\mathfrak{g}$-module. To this end, we must show that for $x, x' \in \mathfrak{g}$ and $v \in V$,

$$\pi\left([x, x'] \cdot v\right) = \pi(x \cdot \pi(x' \cdot v)) - \pi(x' \cdot \pi(x \cdot v)).$$

But by definition, $\pi\left([x, x'] \cdot v\right) = \pi(xx'v) - \pi(x'xv)$. Hence, it suffices to show that

$$\pi\left(x \cdot \pi(x' \cdot v)\right) = \pi(xx'v).$$

We can write $x'v = \pi(x'v) + v'$, where v' is a linear combination of basis elements $\mathbf{x}^k$ with $k_i > 0$ for some $p < i \leqslant n$. So it suffices to show that $\pi(x\mathbf{x}^k) = 0$ for some such k and every $x \in \mathfrak{g}$. For if this is shown, then

$$\pi(xx'v) = \pi\left(x\pi(x'v) + xv'\right) = \pi(x\pi(x'v)) + \pi(xv') = \pi(x\pi(x'v)),$$

as desired.

Therefore, let $x \in \mathfrak{g}$ and k a multivector of nonnegative integers such that $k_i > 0$ for some $p < i \leqslant n$. We can write $\mathbf{x}^k$ as $\mathbf{x}^h\mathbf{x}^l$ where $h_i = 0$ for $p < i \leqslant n$ and $l_i = 0$ for $1 \leqslant i \leqslant p$. Note that $l \neq 0$ as a vector. Then, we may write $x\mathbf{x}^h$ as a linear combination of elements $\mathbf{x}^{h'}\mathbf{x}^{l'}$ with $h'_i = 0$ for $i < p \leqslant n$ and $l'_i = 0$ for $1 \leqslant i \leqslant p$. In conclusion, $x\mathbf{x}^k$ is a linear combination of elements of the form $\mathbf{x}^{h'}\mathbf{x}^{l'}\mathbf{x}^l$.

Since $\mathfrak{h}$ is a subalgebra, note that the vector space spanned by $\mathbf{x}^l$ where $l_i = 0$ for $1 \leqslant i \leqslant p$ is precisely the universal enveloping algebra of $\mathfrak{h}$. In particular, it is an associative k-subalgebra of $\mathfrak{U}$. This shows that $\mathbf{x}^{l'}\mathbf{x}^l$ can be written as a linear combination of elements of the form $\mathbf{x}^{l''}$ where $l''_i = 0$ for $1 \leqslant i \leqslant p$. Since the terms of degree greater than 0 form an ideal in $\mathfrak{U}(\mathfrak{h}) \subseteq \mathfrak{U}$, $l \neq 0$ will imply $l'' \neq 0$.

The above paragraph shows that $x\mathbf{x}^k$ can be written as a linear combination of elements of the form $\mathbf{x}^{h'}\mathbf{x}^{l''}$ where $h'_i = 0$ for $p < i \leqslant n$ and $l''_i = 0$ for $1 \leqslant i \leqslant p$ and $l'' \neq 0$ as a vector. But these are all basis elements and are not contained in V, therefore, map to 0 under π. Hence, $\pi(x\mathbf{x}^k) = 0$ whenever k is such that $k_i > 0$ for some $p < i \leqslant n$.

We have established that V is indeed a $\mathfrak{g}$-module. The element $1 \in V$ is not annihilated by $\mathfrak{g}$, but is annihilated by $\mathfrak{h}$ since for any $y \in \mathfrak{h}$, $y \cdot 1 = \pi(y) = 0$, since $y \notin V \subseteq \mathfrak{U}$. Invoking Lemma 5.2, we have that $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is not an epimorphism, thereby completing the proof. ∎

REMARK 5.4. In the case that $\mathfrak{g}$ is infinite-dimensional, choose first a k-basis $\{x_\beta \colon \beta \in B\}$ of $\mathfrak{h}$ and extend it to a k-basis

$$\{x_\alpha \colon \alpha \in A\} \sqcup \{x_\beta \colon \beta \in B\}$$

of $\mathfrak{g}$. Well order A and B separately and define a well order on $A \sqcup B$ by setting $a < b$ whenever $a \in A$ and $b \in B$. That this is indeed a well-order is easy to check. The proof then remains unchanged by replacing each instance of "$1 \leqslant i \leqslant p$" with "$i \in A$" and "$p < i \leqslant n$ with $i \in B$".

The analogue of Theorem 5.3 is not true in the category of finite-dimensional Lie algebras:

THEOREM 5.5. Let k be algebraically closed and char $k = 0$, $\mathfrak{g} = \mathfrak{sl}_2(k)$, and $\mathfrak{h} \subsetneq \mathfrak{g}$ be the subalgebra of upper triangular matrices in $\mathfrak{g}$. Then, $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is an epimorphism.

Proof. Recall that there is the standard basis $\{h, x, y\}$ of $\mathfrak{g}$ over k and $\mathfrak{h}$ is generated by $\{h, x\}$. For the sake of this proof, make the replacements $h \mapsto \frac{1}{2}h$, $x \mapsto \frac{1}{\sqrt{2}}x$, and $y \mapsto \frac{1}{\sqrt{2}}y$, so that

$$[h, x] = x, \quad [h, y] = -y, \quad \text{and} \quad [x, y] = h.$$

Suppose now that there are two morphisms $\mathfrak{g} \to \mathfrak{t}$ that agree on $\mathfrak{h}$. We shall show that both the morphisms are equal. Since $\mathfrak{g}$ is simple, both morphisms must be injective, unless they are both 0 in which case there's nothing to prove. Denote the images of h and x in $\mathfrak{t}$ by h and x since both morphisms agree here and let y, y' denote the images of y in $\mathfrak{t}$ under the two morphsims and suppose that $y \neq y'$.

Set $u_0 = y - y'$, and define $u_n = [y, u_{n-1}]$ for $n \geqslant 1$ with the convention that $u_{-1} = 0$. Note that $[x, u_0] = [x, y] - [x, y'] = 0$ and $[h, u_0] = -u_0$.

Claim 1. $[x, u_n] = -\frac{1}{2}n(n+1)u_{n-1}$ and $[h, u_n] = -(n+1)u_n$ for $n \geqslant 0$.

We induct on n. The base case of $n = 0$ is clear. For $n \geqslant 1$, we have

$$\begin{aligned} &[x, [y, u_{n-1}]] + [y, [u_{n-1}, x]] + [u_{n-1}, [x, y]] = 0 \\ \Longrightarrow\ &[x, u_n] + \frac{1}{2}n(n-1)[y, u_{n-2}] + [u_{n-1}, h] \\ \Longrightarrow\ &[x, u_n] = -\frac{1}{2}n(n-1)u_{n-1} - nu_{n-1} = -\frac{1}{2}n(n+1)u_{n-1}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &[h, [y, u_{n-1}]] + [y, [u_{n-1}, h]] + [u_{n-1}, [h, y]] = 0 \\ \Longrightarrow\ &[h, u_n] + n[y, u_{n-1}] - [u_{n-1}, y] = 0 \\ \Longrightarrow\ &[h, u_n] = -(n+1)u_n. \end{aligned}$$

This proves Claim 1.

Claim 2. $\{h, x, y\} \cup \{u_n \colon n \geqslant 0\}$ is linearly independent.

Suppose not. Then there is a linear combination

$$\alpha_n u_n + \cdots + \alpha_0 u_0 + \alpha_h h + \alpha_x x + \alpha_y y = 0.$$

with $\alpha_n \neq 0$. If $n \geqslant 1$, then apply $[x, \cdot]$ until you are left with a linear combination of the form

$$\beta_0 u_0 + \beta_h h + \beta_x x + \beta_y y = 0$$

where $\beta_0 \neq 0$. Next, applying $[h, \cdot]$, we have

$$-\beta_0 u_0 + \beta_x x - \beta_y y = 0.$$

Adding the above two equations, we have $2\beta_x x + \beta_h h = 0$, whence $\beta_x = \beta_h = 0$. This gives $\beta_0 u_0 + \beta_y y = 0$. Applying $[x, \cdot]$, we get $\beta_y = 0$, which leaves us with $\beta_0 u_0 = 0$, which is absurd, since $u_0 \neq 0$ and $\beta_0 \neq 0$. This proves Claim 2.

Finally, we have our desired contradiction, since $\mathfrak{t}$ is a finite-dimensional Lie algebra. It follows that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is epimorphic. ■