The Jacobson Radical

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§1 Jacobson Radical

Definition 1.1 (Jacobson radical). Let R be a ring. The *Jacobson radical* of R is defined to be the intersection of all left maximal ideals in R and is denoted by $\operatorname{rad} R$.

Lemma 1.2. For $y \in R$, the following are equivalent:

(1) $y \in \operatorname{rad} R$.

(2) For every $x \in R$, $1 - xy$ is left-invertible in R.

(3) For every simple (left) R-module M, $yM = 0$.

Proof. $1 \implies 2$: If $1 - xy$ were not left-invertible, then it would be contained in a maximal left ideal $\mathfrak{m}$. But $y \in \mathfrak{m}$ and hence, $1 \in \mathfrak{m}$, a contradiction.

$2 \implies 3$: If $yM \neq 0$, then $yM = M$. Also, M is a cyclic module, generated by some $m \in M$. Then, there is some $m' \in M$ such that $m = ym'$. So $M = R(ym')$. Hence, there is some $x \in R$ such that $m' = xym'$, equivalently, $(1 - xy)m' = 0$ whence $m' = 0$ and consequently, $m = 0$.

$3 \implies 1$: Every simple (left) R-module is of the form $R/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal left ideal in R. Therefore, $y \in \mathfrak{m}$ for every maximal left ideal in R. Thus, $y \in \operatorname{rad} R$. ■

Corollary. $\operatorname{rad} R$ is a two-sided ideal of R.

Proof.

$$\operatorname{rad} R = \bigcap \operatorname{Ann}_R(M),$$

where the intersection ranges over representatives from equivalence classes of simple R-modules under R-isomorphism. Recall that $\operatorname{Ann}_R(M)$ is always a two-sided ideal of R. ■

Proposition 1.3. For $y \in R$, the following are equivalent:

(1) $y \in \operatorname{rad} R$.

(2) For all $x, z \in R$, $1 - xyz \in R^\times$.

Proof. $1 \implies 2$: Obviously, $yz \in \operatorname{rad} R$ and hence, $1 - xyz$ is left-invertible. Let $u \in R$ be such that $u(1 - xyz) = 1$, that is, u is right-invertible. Therefore, $u = 1 + uxyz$ whence, u is left-invertible and hence in $R^\times$. It follows that $(1 - xyz)u = 1$ and $1 - xyz \in R^\times$.

$2 \implies 1$: Take $z = 1$. ■

PROPOSITION 1.4. Let $\mathfrak{a} \trianglelefteq R$ be a two-sided ideal contained in $\operatorname{rad} R$. Then, $\operatorname{rad}(R/\mathfrak{a}) = (\operatorname{rad} R)/\mathfrak{a}$.

Proof. A left maximal ideal of $R/\mathfrak{a}$ is $\mathfrak{m}/\mathfrak{a}$ where $\mathfrak{m}$ is a left maximal ideal of R. Conversely if $\mathfrak{m}$ is a left maximal ideal of R, then it contains $\operatorname{rad} R$ and hence, $\mathfrak{a}$. Consequently, $\mathfrak{m}/\mathfrak{a}$ is a left maximal ideal of $R/\mathfrak{a}$. The conclusion follows. ■

COROLLARY. Let $\overline{R} = R/\operatorname{rad} R$. Then $\operatorname{rad} \overline{R} = 0$.

DEFINITION 1.5 (SEMIPRIMITIVE). A ring R is said to be *semiprimitive* or *Jacobson semisimple* if $\operatorname{rad} R = 0$.

DEFINITION 1.6. A one-sided (resp. two-sided) ideal $I \trianglelefteq R$ is said to be *nil* if every element in I is nilpotent. It is said to be *nilpotent* if there is a positive integer $n > 0$ such that $I^n = 0$.

REMARK 1.7. It is immediate from the definition that every nilpotent ideal is nil. The converse is not true. Consider

$$R = k[x_1, x_2, \dots]/(x_1, x_2^2, x_3^3, \dots).$$

The maximal ideal $\overline{\mathfrak{m}} = (\overline{x}_1, \overline{x}_2, \dots)$ is nil but not nilpotent.

PROPOSITION 1.8. Let $I \trianglelefteq_R R$ be a nil left ideal. Then, $I \subseteq \operatorname{rad} R$.

Proof. Let $y \in I$ and $x \in R$. Then $xy \in I$ is nilpotent. Consequently, $1 - xy$ is a unit. ■

LEMMA 1.9 (NAKAYAMA-AZUMAYA-KRULL). For any left ideal $J \trianglelefteq_R R$, the following are equivalent:

(1) $J \subseteq \operatorname{rad} R$.

(2) For any finitely generated (left) R-module M, $JM = M$ implies $M = 0$.

(3) For any (left) R-modules $N \subseteq M$ such that M/N is finitely generated, $N + JM = 0$ implies $N = M$.

Proof. $1 \implies 2$: Suppose $M \neq 0$. Pick a minimal set of generators $\{m_1, \dots, m_n\} \subseteq M$. Then, $m_n = a_1 m_1 + \dots a_n m_n$, consequently,

$$(1 - a_n)m_n = a_1 m_1 + \cdots + a_{n-1} m_{n-1}.$$

But $1 - a_n$ is a unit and hence, m_n can be expressed as a linear combination of $\{m_1, \dots, m_{n-1}\}$ contradicting the minimality of the set of generators.

$2 \implies 3$: Consider M/N.

$3 \implies 1$: Suppose there is some $y \in J \setminus \operatorname{rad} R$. Then, there is a left maximal ideal $\mathfrak{m}$ that does not contain y. As a result, $\mathfrak{m} + J \cdot R = R$, implying that $\mathfrak{m} = R$, which is absurd. This completes the proof. ■

PROPOSITION 1.10. Let R be left artinian. Then, $\operatorname{rad} R$ is nilpotent, consequently, it is the largest nilpotent left (resp. right) ideal.

Proof. Let $J = \operatorname{rad} R$. There is a descending chain of left ideals,

$$J \supseteq J^2 \supseteq \dots$$

which must stabilize. Let $I = J^n = J^{n+1} = \cdots$, which is a left ideal. Suppose $I \neq 0$. Let $\Sigma = \{\mathfrak{a} \trianglelefteq R \mid I\mathfrak{a} \neq 0\}$. This is non-empty for it contains I. Let $\mathfrak{a}_0$ be a minimum element in Σ. Then, there is some $a \in \mathfrak{a}_0$ such that $Ia \neq 0$. Consequently, $\mathfrak{a}_0 = Ra$. On the other hand, note that $I(I\mathfrak{a}_0) = I^2\mathfrak{a}_0 = I\mathfrak{a}_0$, whereby $I\mathfrak{a}_0 \in \Sigma$ and $I\mathfrak{a}_0 = \mathfrak{a}_0$. Nakayama's lemma (Lemma 1.9) implies $\mathfrak{a}_0 = 0$, a contradiction. Thus, $I = 0$ and $\operatorname{rad} R$ is nilpotent. ■

THEOREM 1.11. For a ring R, the following are equivalent:

(1) R is semisimple.

(2) R is semiprimitive and left artinian.

Proof. $1 \implies 2$: Note that ${}_RR$ is a finite direct sum of minimal left-ideals, which are artinian modules over R. Therefore, ${}_RR$ is a left artinian.

We shall now show that $\operatorname{rad} R = 0$. Indeed, let $\mathfrak{a} = \operatorname{rad} R$. Then, there is a left ideal $\mathfrak{b}$ such that $R = \mathfrak{a} \oplus \mathfrak{b}$. Then, there are idempotents e, f such that $e + f = 1$ and $\mathfrak{a} = Re$ and $\mathfrak{b} = Rf$. Note that $f = 1 - e$ and hence, a unit, whence $\mathfrak{b} = (1)$ and $\mathfrak{a} = 0$.

$2 \implies 1$: We shall show that ${}_RR$ is a semisimple module. Pick a minimal left ideal $\mathfrak{a}_1$. Then, there is a maximal left ideal $\mathfrak{m}_1$ such that $\mathfrak{a}_1 \not\subseteq \mathfrak{m}_1$ and hence, $R = \mathfrak{a}_1 \oplus \mathfrak{m}_1$. Set $\mathfrak{b}_1 = \mathfrak{m}_1$. Now, if $\mathfrak{b}_1$ is non-zero, then it contains a minimal left-ideal $\mathfrak{a}_2$. Then, there is a maximal ideal $\mathfrak{m}_2$ such that $R = \mathfrak{a}_2 \oplus \mathfrak{m}_2$. It then follows that $\mathfrak{b}_1 = \mathfrak{a}_2 \oplus (\mathfrak{b}_1 \cap \mathfrak{m}_2)$. Set $\mathfrak{b}_2 = \mathfrak{b}_1 \cap \mathfrak{m}_2$ and continue this way.

Then, we obtain a strictly descending chain

$$\mathfrak{b}_1 \supsetneq \mathfrak{b}_2 \supsetneq \cdots .$$

This must stabilize and when it does, it must stabilize at 0. This gives us a decomposition of ${}_RR$ in terms of minimal left ideals and the proof is complete. ■

COROLLARY (CONVERSE OF MASCHKE). Let k be a field with $\operatorname{char} k = p > 0$. Let G be a finite group such that $p \mid |G|$. Then, kG is not semisimple.

Proof. Let $\sigma = \sum_{g \in G} g$. Then, $k\sigma$ is a two-sided ideal of kG. Further, $\sigma^2 = 0$. Consequently, $k\sigma \subseteq \operatorname{rad} kG$, whence kG is not semisimple. ■

PROPOSITION 1.12. Let R be a ring. Then $\operatorname{rad} M_n(R) = M_n(\operatorname{rad} R)$.

Proof. First, we shall show that $M_n(\operatorname{rad} R) \subseteq \operatorname{rad} M_n(R)$. To do so, it suffices to show that $xE_{ij} \in \operatorname{rad} M_n(R)$ whenever $x \in \operatorname{rad} R$ and $1 \leqslant i, j \leqslant n$. Let $A \in M_n(R)$ be given by $A = (a_{kl})_{1 \leqslant k,l \leqslant n}$. Then,

$$I - AxE_{ij} = I - \sum_{k=1}^{n} a_{ki}xE_{kj} = \underbrace{I - a_{ji}E_{jj}}_{B} - \underbrace{\sum_{k \neq j} a_{ki}E_{kj}}_{N} .$$

Note that B is a unit and N is nilpotent. Therefore, $B - N$ is a unit. This shows that $xE_{ij} \in \operatorname{rad} M_n(R)$, whence $M_n(\operatorname{rad} R) \subseteq \operatorname{rad} M_n(R)$.

Conversely, note that $\operatorname{rad} M_n(R) = M_n(\mathfrak{a})$ for some two-sided ideal $\mathfrak{a} \trianglelefteq R$. This implies that for every $x \in R$ and $a \in \mathfrak{a}$, $I - xaE_{11}$ is invertible. Consequently, $1 - xa$ must be invertible in R and as a result, $a \in \operatorname{rad} R$. Hence, $M_n(\mathfrak{a}) \subseteq M_n(\operatorname{rad} R)$. This completes the proof. ■

THEOREM 1.13 (HOPKINS-LEVITZKI). Let R be a semiprimary ring and M a left R-module. Then the following are equivalent:

(1) M is noetherian.

(2) M is artinian.

Proof. Let $J = \operatorname{rad} R$. Then, there is a positive integer $n > 0$ such that $J^n = 0$. This gives a filtration

$$M \supseteq JM \supseteq \cdots \supseteq J^{n-1}M \supseteq J^nM = 0.$$

The successive quotients $J^iM/J^{i+1}M$ is a $\overline{R}$-module whence it is artinian if and only if it is noetherian. Induct using the exact sequence

$$0 \longrightarrow J^{i+1}M \longrightarrow J^iM \longrightarrow J^iM/J^{i+1}M \longrightarrow 0.$$

This completes the proof. ■

COROLLARY. A left artinian ring is left noetherian.

Proof. A left artinian ring is semiprimary. ■

§2 VON NEUMANN REGULAR RINGS

LEMMA 2.1. If a left ideal $\mathfrak{a} \trianglelefteq {}_RR$ is a direct summand of R, then it is generated by an idempotent.

Proof. There is a left ideal $\mathfrak{b}$ such that $R = \mathfrak{a} \oplus \mathfrak{b}$ as left ideals. Hence, $1 = e + f$ for some $e \in \mathfrak{a}$ and $f \in \mathfrak{b}$. Then, $e = e \cdot 1 = e(e + f) = e^2 + ef$. Note that this means $ef \in \mathfrak{a}$ but $ef \in \mathfrak{b}$ (because $\mathfrak{b}$ is a left ideal) and hence, $ef \in \mathfrak{a} \cap \mathfrak{b} = 0$. Consequently, $e = e^2$ is an idempotent. Now, for any $a \in \mathfrak{a}$, $a = ae + af = ae$ because $af \in \mathfrak{b}$ and $af = a - ae \in \mathfrak{a}$ whence $af = 0$. ■

THEOREM 2.2. For a ring R, the following are equivalent:

(1) For any $a \in R$, there is an $x \in R$ such that $a = axa$.

(2) Every principal left ideal is generated by an idempotent.

(3) Every principal left ideal is a direct summand of ${}_RR$.

(4) Every finitely generated left ideal is generated by an idempotent.

(5) Every finitely generated left ideal is a direct summand of ${}_RR$.

A ring satisfying any one of the above five equivalent conditions is called a *von Neumann regular ring*.

Proof. First, we show equivalences $2 \iff 3$ and $4 \iff 5$.

$2 \iff 3$: Let $e \in R$ be an idempotent. Then, $R = Re \oplus R(1-e)$. The converse follows from Lemma 2.1.

$4 \iff 5$: The forward implication follows in the same way as $2 \implies 3$ and the converse follows from Lemma 2.1.

$1 \implies 2$: Let Ra be a principal left ideal in R for some $a \in R$. Then, there is an $x \in R$ such that $a = axa$, consequently, $xa = xaxa$. Set $e = xa$. Then, $e = e^2$ whence e is an idempotent. Further, note that $Re \subseteq Ra$ and $a = ae \in Re$ whereby $Ra = Re$.

$2 \implies 1$: Let Ra be a principal left ideal in R for some $a \in R$. Then, there is an idempotent $e \in R$ such that $Ra = Re$. There are $x, y \in R$ such that $e = xa$ and $a = ye$. Now,

$$a = ye = ye^2 = axa.$$

$4 \implies 2$: Clear.

$2 \implies 4$: To see this direction, it suffices to show that a left ideal generated by two idempotents is generated by a single idempotent. Indeed, let $\mathfrak{a} = R(e, f)$ where e and f are idempotents in R. Note that $\mathfrak{a} = Re + Rf(1-e)$. Since $Rf(1-e)$ is a principal left ideal, it is generated by an idempotent Re'. Note that $e'e = xf(1-e)e = 0$ for some $x \in R$.

Let $g = 1 - (1-e)(1-e') = e + e' - ee'$. Note that g is an idempotent and $g \in Re + Re'$. Further, $eg = e$ and $e'g = e'$ whereby $Rg = Re + Re'$. This completes the proof. ■

COROLLARY. Semisimple $\implies$ von Neumann regular $\implies$ semiprimitive.

Proof. The first implication is clear. Suppose R is von Neumann regular. Let $a \in \operatorname{rad} R$. Then, there is $x \in R$ such that $a = axa$, that is, $a(1-xa) = 0$. But $1 - xa$ is a unit in R and hence, $a = 0$. Thus, R is semiprimitive. ■

THEOREM 2.3. Left noetherian + von Neumann regular $\implies$ semisimple.

Proof. Let $\mathfrak{a} \trianglelefteq {}_RR$ be a left ideal. Since R is left noetherian, $\mathfrak{a}$ is finitely generated and hence, a direct summand of ${}_RR$. As a result, ${}_RR$ is semisimple. ■

COROLLARY. Left noetherian + von Neumann regular $\implies$ left artinian.