# The Inverse Galois Problem over $\mathbb{C}(t)$

### Swayam Chube

December 1, 2024

#### **Abstract**

This is an attempt to present a self-contained proof of the Inverse Galois Problem over  $\mathbb{C}(t)$ . The only result used without proof is Riemann's Existence Theorem (Theorem 3.1).

## §1 RIEMANN SURFACES AND HOLOMORPHIC MAPS

**DEFINITION 1.1.** Let X be a two-dimensional manifold. A *complex chart* on X is a homeomorphism  $\varphi: U \to V$  of an open subset  $U \subseteq X$  onto an open subset  $V \subseteq \mathbb{C}$ .

Two complex charts  $\varphi_i: U_i \to V_i$ , i=1,2 are said to be *holomorphically compatible* if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is biholomorphic.

A *complex atlas* on X is a system  $\mathfrak{A} = \{ \varphi_i : U_i \to V_i \mid i \in I \}$  of charts which are holomorphically compatible and which cover X. Two complex atlases  $\mathfrak{A}$  and  $\mathfrak{A}'$  on X are said to be *analytically equivalent* if every chart of  $\mathfrak{A}$  is holomorphically compatible with every chart of  $\mathfrak{A}'$ .

A *complex structure* on a two-dimensional manifold X is an equivalence class of analytically equivalent atlases on X.

A *Riemann surface* is a pair  $(X, \Sigma)$  where X is a connected two-dimensional manifold and  $\Sigma$  is a complex structure on X.

**EXAMPLE 1.2.** Consider  $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$ , the one-point compactification of  $\mathbb{C}$ . Let  $U_1 = \mathbb{C} \subseteq \mathbb{P}^1$  and  $U_2 = \mathbb{C}^* \cup \{\infty\}$ . Consider the charts  $\varphi_1 : U_1 \to \mathbb{C}$ , the identity map, and  $\varphi_2 : U_2 \to \mathbb{C}$  given by

$$\varphi(z) = \begin{cases} \frac{1}{z} & z \in \mathbb{C}^* \\ 0 & z = \infty. \end{cases}$$

These are compatible charts since the transition function is  $z \mapsto \frac{1}{z}$  on  $\mathbb{C}^*$ .

**EXAMPLE 1.3.** If X is a Riemann surface and  $Y \subseteq X$  is a connected open set, then every chart of X restricts to a chart on Y (by restriction of the domain) and these are still holomorphically comptible. Thus, Y inherits a natural Riemann surface structure from X. In particular, every open subset of  $\mathbb{C}$  is a Riemann surface.

**DEFINITION 1.4.** A map  $f: X \to Y$  of Riemann surfaces is said to be *holomorphic* if for every pair of charts  $\psi_1: U_1 \to V_1$  on X and  $\psi_2: U_2 \to V_2$  on Y with  $f(U_1) \subseteq U_2$ , the mapping  $\psi_2 \circ f \circ \psi_1^{-1}: V_1 \to V_2$  is holomorphic.

A holomorphic function on X means a holomorphic function  $f: X \to \mathbb{C}$ . These form a ring denoted by  $\mathcal{O}(X)$ .

THEOREM 1.5 (RIEMANN'S REMOVABLE SINGULARITIES THEOREM). Let X be a Riemann surface and  $f \in \mathcal{O}(X \setminus \{a\})$ . If f is bounded in a neighborhood of a, then f can be extended to a holomorphic function  $\widetilde{f} \in \mathcal{O}(X)$ .

*Proof.* Follows from the analogous statement in elementary complex analysis.

**THEOREM 1.6 (IDENTITY THEOREM).** Let X and Y be Riemann surfaces and  $f_1, f_2 : X \to Y$  be two holomorphic mappings which coincide on a set  $A \subseteq X$  having a limit point  $a \in X$ . Then  $f_1 = f_2$ .

Proof. Let

 $B = \{x \in X : \text{ there is a neighborhood } W \text{ of } x \text{ such that } f_1|_W = f_2|_W \}.$ 

By definition, B is open. By continuity, note that  $a \in A$ . Considering charts centered at a and  $f_1(a) = f_2(a)$ , and using the identity theorem from elementary complex analysis, it is not hard to see that  $a \in B$ , that is,  $B \neq \emptyset$ . Finally, suppose  $b_n \to b \in X$ . Then, by continuity,  $y = f_1(b) = f_2(b)$ . Consider charts centered at b and b. Note that b lies in the chart centred at b for sufficiently large b and hence, it would follow that  $b \in B$ . Thus, b is a clopen nonempty subset of b. Owing to the connectedness of b. This completes the proof.

**DEFINITION 1.7.** A *meromorphic function* on a Riemann surface X is a holomorphic function  $f: X' \to \mathbb{C}$ , where  $X' \subseteq X$  is an open subset, such that the following hold:

- (a)  $X \setminus X'$  is discrete.
- (b) For every  $p \in X \setminus X'$ ,

$$\lim_{x \to p} |f(x)| = \infty.$$

The points of  $X \setminus X'$  are called the *poles* of f. The set of all meromorphic functions on X is denoted by  $\mathcal{M}(X)$ .

**PROPOSITION 1.8.** There is a canonical correspondence between  $\mathcal{M}(X)$  and the set of holomorphic functions  $X \to \mathbb{P}^1$ .

Proof. Straightforward.

**COROLLARY.**  $\mathcal{M}(X)$  is a field.

#### §§ Local Normal Form

**THEOREM 1.9.** Let X and Y be Riemann surfaces and  $f: X \to Y$  a non-constant holomorphic map. Suppose  $a \in X$  and  $b = f(a) \in Y$ . Then, there exists an integer  $k \geqslant 1$  and charts  $\varphi: U \to V$  on X and  $\psi: U' \to V'$  on Y with the following properties:

- (i)  $a \in U$ ,  $\varphi(a) = 0$ ,  $b \in U'$  and  $\psi'(b) = 0$ .
- (ii)  $f(U) \subseteq U'$ .
- (iii) The diagram

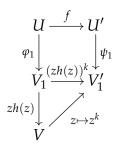
$$U \longrightarrow U'$$

$$\varphi \downarrow \qquad \qquad \downarrow \psi$$

$$V \xrightarrow[z\mapsto z^k]{} V'$$

commutes. The number k is called the *multiplicity* of f at a.

*Proof.* Begin with two charts  $\varphi_1: U \to V_1$  and  $\psi_1: U' \to V'_1$  satisfying (i) and (ii). The induced map  $V_1 \to V'_1$  takes 0 to 0 and hence, is of the form  $z^k g(z)$  for some  $k \geqslant 1$  and holomorphic  $g: V_1 \to V'_1$  with  $g(0) \neq 0$ . Shrinking all the open sets if necessary, we may suppose that  $g(z) = h(z)^k$  for some holomorphic function  $h: V_1 \to \mathbb{C}$ . Note that zh(z) must be injective and non-constant on  $V_1$  whence maps  $V_1$  biholomorphically onto some  $V \subseteq \mathbb{C}$ . We obtain the following commutative diagram



thereby completing the proof.

**THEOREM 1.10 (OPEN MAPPING THEOREM).** A non-constant holomorphic map between Riemann surfaces is open.

*Proof.* Since being open is a local property, this follows immediately from Theorem 1.9.

**COROLLARY.** Let  $f: X \to Y$  be an injective holomorphic map of Riemann surfaces. Then f is a biholomorphic mapping of X onto Z = f(X).

*Proof.* Due to Theorem 1.10,  $Z \subseteq Y$  is open. Since f is injective, it follows from Theorem 1.9 that k = 1 at each point of X. In particular, f is a local homeomorphism onto Z. The conclusion follows.

**THEOREM 1.11.** If X is a compact Riemann surface and  $f: X \to Y$  a non-constant holomorphic map of Riemann surfaces, then f is surjective.

*Proof.* The image of f is both open and closed in Y.

**COROLLARY.** If *X* is a compact Riemann surface, then  $\mathcal{O}(X)$  consists of only constant functions.

#### §§ Branched and Unbranched Coverings

**DEFINITION 1.12.** Let  $p: Y \to X$  be a non-constant holomorphic map of Riemann surfaces. A point  $y \in Y$  is said to be a *branch point* or *ramification point* of p, if there is no neighborhood V of y such that  $p|_V$  is injective. The map p is called an *unbranched holomorphic map* if it has no branch points.

**THEOREM 1.13.** A non-constant holomorphic map  $p: Y \to X$  is unbranched if and only if p is a local homeomorphism, i.e., every point  $y \in Y$  has an open neighborhood V which is mapped homeomorphically by p onto an open set U in X.

*Proof.* Immediate from the definition since an injective map of Riemann surfaces is a biholomorphism onto its image.

**THEOREM 1.14.** Let X be a Riemann surface, Y a connected Hausdorff topological space, and  $p: Y \to X$  a local homeomorphism. Then there is a unique complex structure on Y such that p is holomorphic.

*Proof.* Suppose  $\varphi_1: U_1 \to V_1 \subseteq \mathbb{C}$  is a chart of the complex structure of X such that there is an open subset  $U \subseteq Y$  with  $p|_U: U \to U_1$  a homeomorphism. Then,  $\varphi:=\varphi \circ p: U \to V$  is a complex chart on Y. Let  $\mathfrak A$  be the set of all complex charts on Y obtained in this way. It is easy to see that the charts of  $\mathfrak A$  cover Y and are holomorphically compatible. Thus, we have defined a complex structure on Y and it follows that p is a holomorphic map when Y is equipped with this structure.

Suppose  $(Y, \Sigma)$  and  $(Y, \Sigma')$  are two complex charts such that p is holomorphic, then  $id : (Y, \Sigma) \to (Y, \Sigma')$  is a bijective holomorphic map, whence a biholomorphism. This shows uniqueness.

**THEOREM 1.15.** Let X, Y, Z be Riemann surfaces,  $p: Y \to X$  an unbranched holomorphic map and  $f: Z \to X$  any holomorphic map. Then, every continuous lift  $g: Z \to Y$  of f is holomorphic.

*Proof.* Let  $z \in Z$ , x = f(z), and y = g(z). There is a neighborhood V of y in Y such that  $p|_V$  is injective. Let  $U = p(V) \subseteq X$ , which is open and biholomorphic to V through p. If  $W = g^{-1}(V)$ , then  $g|_W = p|_V^{-1} \circ f|_W$  whence g is holomorphic.

**DEFINITION 1.16.** A continuous map  $f: X \to Y$  of topological spaces is said to be *proper* if  $f^{-1}(K)$  is compact in X for every compact subset K of Y. The map f is said to be *discrete* if every fiber is discrete in X.

**LEMMA 1.17.** A proper map between locally compact Hausdorff spaces is closed.

*Proof.* Follows from the fact that a subset of an LCH space is closed if and only if its intersection with every compact subset is closed.

**COROLLARY.** A proper holomorphic map between Riemann surfaces is surjective.

*Proof.* The image is both closed and open.

**LEMMA 1.18.** Let *X* and *Y* be locally compact Hausdorff. If  $p: Y \to X$  is a proper, discrete map then:

- (a) for every  $x \in X$ , the set  $p^{-1}(x)$  is finite.
- (b) if  $x \in X$  and V is a neighborhood of  $p^{-1}(x)$ , then there is a neighborhood U of x with  $p^{-1}(U) \subseteq V$ .

*Proof.* (a) Compact discrete sets must be finite.

(b) Since  $Y \setminus V$  is closed, due to the preceding lemma,  $A = p(Y \setminus V)$  is closed in X and  $x \notin A$ . Hence,  $U = X \setminus A$  is an open neighborhood of x such that  $p^{-1}(U) \subseteq V$ .

**THEOREM 1.19.** Let X and Y be locally compact Hausdorff spaces and  $p: Y \to X$  a proper local homeomorphism. Then p is a covering map.

*Proof.* Choose any  $x \in X$  and let  $p^{-1}(x) = \{y_1, \dots, y_n\}$ . Since p is a local homeomorphism, we can inductively choose disjoint neighborhoods  $W_i$  of  $y_i$  and a neighborhood V of x such that the restriction  $p|_{W_i}: W_i \to V$  is a homeomorphism. It follows that p is a covering map.

**PROPOSITION 1.20.** The set of branch points of a non-constant holomorphic map between Riemann surfaces is a discrete closed set.

*Proof.* Let  $f: X \to Y$  be a non-constant holomorphic map. Let  $a \in X$  be a branch point and b = f(a). Then due to Theorem 1.9, there are charts  $\varphi: U \to V$  and  $\varphi': U' \to V'$  centered at a and b respectively such that the induced map  $V \to V'$  is  $z \mapsto z^k$  for some positive integer  $k \ge 2$  (since a is a branch point). But for any  $0 \ne z \in V'$ , the map  $V \to V'$  is a local homeomorphism and hence, the set of branch points forms a discrete set.

To see that it is closed, let  $a \in X$  not be a branch point. Then, there is a neighborhood V of a on which f is injective and hence, none of the points in V are branch points. This shows that the set of branch points is also closed.

**DEFINITION 1.21.** Let  $f: X \to Y$  be a proper holomorphic map. As we have seen earlier, f is surjective. Let  $A \subseteq X$  be the set of branch points of f. Since f is proper, the set  $B = f(A) \subseteq Y$  is closed and discrete (use the Local Normal Form). One calls B the set of *critical values* of f.

With notation as above, let  $Y' = Y \setminus B$  and  $X' = f^{-1}(Y') \subseteq X \setminus A$ . The restriction  $f: X' \to Y'$  is a proper unbranched holomorphic covering map since it is a local homeomorphism (owing to the fact that all branch points have been removed). It has a well-defined finite number of sheets, say n. Thus, every value  $c \in Y'$  is taken precisely n times. We would like to extend this notion to critical values.

For  $x \in X$ , denote by V(f, x), the multiplicity of f at x in the sense of Theorem 1.9. We say that f takes the value  $c \in Y$ , counting multiplicities, m times on X, if

$$m = \sum_{x \in f^{-1}(c)} v(f, x).$$

**THEOREM 1.22.** Let  $f: X \to Y$  be a proper non-constant holomorphic map between Riemann surfaces. Then there exists a natural number n such that f atkes every value  $c \in Y$ , counting multiplicities, n times.

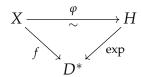
*Proof.* Using the notation as in the preceding paragraph, let n be the number of sheets of the unbranched covering  $f: X' \to Y'$ . Suppose  $b \in B$  is a critical value,  $p^{-1}(b) = \{x_1, \ldots, x_r\}$  and  $k_i = v(f, x_i)$ . Due to Theorem 1.9, there are disjoint neighborhoods  $U_j$  of  $x_j$  and  $y_j$  of  $y_j$  such that for every  $y_j \in V_j \setminus \{y_j\}$  the set  $y_j \in V_j \cap V_j$  consists of exactly  $y_j \in V_j$  points. Due to Lemma 1.18, we can find a neighborhood  $y_j \in V_j \cap \cdots \cap V_r$  of  $y_j \in V_j \cap V_j$  of  $y_j \in V_j \cap V_j$  of  $y_j \in V_j$  of  $y_j$ 

**REMARK 1.23.** A proper non-constant holomorphic map between Riemann surfaces will be called an *n-sheeted holomorphic covering map*, where n is the integer found in the above result. Note that holomorphic covering maps are allowed to have branch points.

Let *D* denote the unit disk in  $\mathbb{C}$  and  $D^* = D \setminus \{0\}$ .

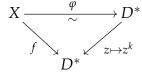
**THEOREM 1.24.** Let  $f: X \to D^*$  be an unbranched holomorphic covering map. Then one of the following holds:

(a) If the covering has an infinite number of sheets, then there exists a biholomorphic mapping  $\varphi: X \to H$  of X onto the left half plane such that



commutes.

(b) If the covering is k-sheeted with  $k < \infty$ , then there exists a biholomorphic mapping  $\varphi: X \to D^*$  such that



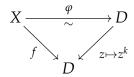
commutes.

*Proof.* Follows from the Galois theory of covers and the fact that H is the universal cover of  $D^*$  and

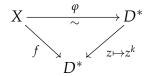
$$Deck(H/D^*) = \{\tau_n \colon n \in \mathbb{Z}\},\$$

where  $\tau_n(z) = z + 2n\pi i$ .

**THEOREM 1.25.** Let  $f: X \to D$  be a proper non-constant holomorphic map which is unbranched over  $D^* = D \setminus \{0\}$ . Then there is a natural number  $k \geqslant 1$  and a biholomorphic map  $\varphi: X \to D$  such that



*Proof.* The preceding theorem furnishes a  $k \ge 1$  making



commute. Let  $p_k : D \to D$  denote the map  $z \mapsto z^k$ . If we show that  $f^{-1}(0)$  is a singleton, then we would be done since we could extend  $\varphi : X \to D$  making the required diagram commute.

Suppose  $f^{-1}(0)$  consists of n points  $b_1, \ldots, b_n$ , where  $n \ge 1$ . Then due to Lemma 1.18 there are disjoint open neighborhoods  $V_i$  of  $b_i$  and a disk  $D(r) = \{z \in \mathbb{C} : |z| < r\}$ ,  $0 < r \le 1$  such that

$$f^{-1}(D(r)) \subseteq V_1 \cup \cdots \cup V_n$$
.

Let  $D^*(r) = D(r) \setminus \{0\}$ . Since  $f^{-1}(D^*(r))$  is homeomorphic to  $p_k^{-1}(D^*(r)) = D^*(\sqrt[k]{r})$ , it is connected. Since every point  $b_i$  is in the closure of  $f^{-1}(D^*(r))$ ,  $f^{-1}(D(r))$  is also connected. Hence, n = 1. This completes the proof.

## §2 ALGEBRAIC FUNCTIONS

**DEFINITION 2.1.** Let  $\pi: Y \to X$  be an n-sheeted unbranched holomorphic covering of Riemann surfaces and  $f \in \mathcal{M}(Y)$ . Every point  $x \in X$  has an open neighborhood U such that  $\pi^{-1}(U)$  is the disjoint union of open sets  $V_1, \ldots, V_n$  and  $\pi: V_v \to U$  is biholomorphic for  $v = 1, \ldots, n$ . Let  $\tau_v : U \to V_v$  denote the inverse of the restricted map  $\pi: V_v \to U$  and let  $f_v = \tau_v^* f := f \circ \tau_v \in \mathcal{M}(U)$ .

Define the *elementary symmetric functions*  $c_1, \ldots, c_n \in \mathcal{M}(U)$  as

$$c_v = (-1)^v \sigma_v \left( f_1, \ldots, f_n \right),$$

where  $\sigma_v$  is the v-th elementary symmetric polynomial in n indeterminates.

This same construction can be carried out about every point in X and it is hard to not see that the elementary symmetric functions glue to global meromorphic functions in  $\mathcal{M}(X)$ . These are known as the *elementary symmetric functions corresponding to f*.

**THEOREM 2.2.** Let  $\pi: Y \to X$  be an n-sheeted branched holomorphic covering map. Suppose  $A \subseteq X$  is a closed discrete subset containing all the critical values of  $\pi$  and let  $B = \pi^{-1}(A)$ . Suppose f is a holomorphic (resp. meromorphic) function on  $Y \setminus B$  and

 $c_1, \ldots, c_n \in \mathcal{O}(X \setminus A)$  (resp.  $\in \mathcal{M}(X \setminus A)$ ) are the elementary symmetric functions of f. Then f may be continued holomorphically (resp. meromorphically) to Y precisely if all the  $c_v$  may be continued holomorphically (resp. meromorphically) to X.

*Proof.* Suppose  $a \in A$  and  $b_1, \ldots, b_m$  are the preimages of a. Suppose (U, z) is a relatively compact coordinate neighborhood centered at a and  $U \cap A = \{a\}$ . Note that  $V \subseteq \overline{V} \subseteq \pi^{-1}(\overline{U})$ , which is compact since  $\pi$  is proper. It follows that V is relatively compact and contains all the  $b_u$ 's.

### **Case 1.** Suppose $f \in \mathcal{O}(Y \setminus B)$

- (a) Suppose f can be continued holomorphically to all the points  $b_{\mu}$ . Then f is bounded on V and hence, on  $V \setminus \{b_1, \ldots, b_m\}$ . This implies that all the  $c_v$ 's are bounded on  $U \setminus \{a\}$ . Thus by Riemann's theorem on removable singularities, they may all be continued holomorphically to a.
- (b) Suppose all the  $c_v$  can be continued holomorphically to a; then they are all bounded on  $U \setminus \{a\}$ . Note that for any  $y \in V \setminus \{b_1, \ldots, b_m\}$ , if  $x = \pi(y)$ , then f(y) is a root of the polynomial

$$T^{n} + c_{1}(x)T^{n-1} + \cdots + c_{n}(x),$$

whose coefficients are uniformly bounded, whence f is bounded in a neighborhood of every  $b_{\mu}$  and hence, can be continued there.

#### **Case 2.** Now suppose $f \in \mathcal{M}(Y \setminus B)$ .

- (a) Assume first that f can be continued meromorphically to all points  $b_{\mu}$ . The function  $\varphi = \pi^*z = z \circ \pi \in \mathcal{O}(V)$  vanishes at all the points  $b_{\mu}$ . Thus,  $\varphi^k f$  may be continued holomorphically to all the points  $b_{\mu}$  if k is sufficiently large. The elementary symmetric functions of  $\varphi^k f$  are  $z^{kv}c_v$  and by the first part of the proof, they may be continued holomorphically to a. Thus, all the  $c_v$  may be continued meromorphically to a.
- (b) Suppose now that all the  $c_v$  can be continued meromorphically to a. There is a sufficiently large k such that all the  $z^{kv}c_v$  can be continued holomorphically to a. Thus due to the first case,  $\varphi^k f$  admits a holomorphic continuation to all the points  $b_\mu$ . This completes the proof.

**THEOREM 2.3.** Let  $\pi: Y \to X$  be a branched holomorphic n-sheeted covering map. If  $f \in \mathcal{M}(Y)$  and  $c_1, \ldots, c_n \in \mathcal{M}(X)$  are the elementary symmetric functions of f, then

$$f^{n} + (\pi^{*}c_{1})f^{n-1} + \dots + (\pi^{*}c_{n-1})f + \pi^{*}c_{n} = 0.$$

- The morphism  $\pi^* : \mathcal{M}(X) \hookrightarrow \mathcal{M}(Y)$  is an algebraic field extension of degree  $\leq n$ .
- Moreover, if there exists an  $f \in \mathcal{M}(X)$  and an  $x \in X$  with preimages  $y_1, \ldots, y_n \in Y$  such that the values  $f(y_v)$  for  $v = 1, \ldots, n$  are all distinct, then the field extension  $\pi^* : \mathcal{M}(X) \hookrightarrow \mathcal{M}(Y)$  has degree n.

*Proof.* The fact that f solves the equation follows immediately from the definition of the elementary symmetric functions. Let  $L = \mathcal{M}(Y)$  and  $K = \mathcal{M}(X)$ . Choose  $f_0 \in L$  maximizing  $n_0 = [K(f_0) : K] \le n$ . Let  $f \in L$  be arbitrary. Then,  $K(f_0, f)$  is a finite extension of K and hence, is of the form  $K(g_0)$  due to the Primitive Element Theorem. But then

$$n_0 \geqslant [K(g_0) : K] = [K(f_0, f) : K] \geqslant [K(f_0) : K] = n_0,$$

whence  $f \in K(f_0)$ , that is,  $K(f_0) = L$  and hence,  $[L : K] = n_0 \le n$ .

Now, consider *f* as in the second part of the theorem and suppose its minimal polynomial over *K* looks like

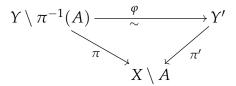
$$f^{m} + (\pi^{*}d_{1})f^{m-1} + \cdots + (\pi^{*}d_{m}) = 0,$$

where  $d_1, \ldots, d_m \in K$ . Under  $\pi, y_1, \ldots, y_n$  map to x and hence,

$$f(y_i)^m + d_1(x)f(y_i)^{m-1} + \dots + d_m(x) = 0,$$

but since the  $f(y_i)$ 's are distinct, we must have  $m \ge n$ , and hence, m = n. This completes the proof.

**THEOREM 2.4.** Suppose X is a Riemann surface,  $A \subseteq X$  is a closed discrete subset and let  $X' = X \setminus A$ . Suppose Y' is another Riemann surface and  $\pi' : Y' \to X'$  a proper *unbranched* holomorphic covering. Then  $\pi'$  extends to a branched covering of X, i.e., there exists a Riemann surface Y, a proper holomorphic mapping  $\pi: Y \to X$  and a biholomorphic mapping  $\varphi: Y \setminus \pi^{-1}(A) \to Y'$  making the diagram



*Proof.* For every  $a \in A$ , choose a coordinate neighborhood  $(U_a, z_a)$  on X such that  $z_a(a) = 0$ ,  $z_a(U_a)$  is the unit disk in  $\mathbb{C}$  and  $U_a \cap U_a' = \emptyset$  if  $a \neq a'$ . Let  $U_a^* = U_a \setminus \{a\}$ . Since  $\pi' : Y' \to X'$  is proper,  $\pi'^{-1}(U_a^*)$  consists of a finite number of connected components  $V_{av}^*$ ,  $v = 1, \ldots, n(a)$ .

For every v, the restricted mapping  $\pi': V_{av}^* \to U_a^*$  is an unbranched covering. Let its covering number be  $k_{av}$ . Due to Theorem 1.24 there are biholomorphic maps  $\zeta_{av}: V_{av}^* \to D^*$  such that

$$V_{av}^* \xrightarrow{\zeta_{av}} D^*$$

$$\pi' \downarrow \qquad \qquad \downarrow \pi_{av}$$

$$U_a^* \xrightarrow{z_a} D^*$$

commutes.

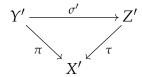
Next, let  $p_{av}$  for  $a \in A$  and v = 1, ..., n(a) be fresh points disjoint from Y' and set

$$Y = Y' \cup \{p_{av} : a \in A, v = 1, ..., n(a)\}.$$

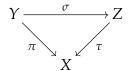
We now topologize Y. If  $W_i$ ,  $i \in I$  is a neighborhood basis of a, then  $\{p_{av}\} \cup (\pi'^{-1}(W_i) \cap V_{av}^*)$ ,  $i \in I$  is set as a neighborhood basis of  $p_{av}$  along with the fact that Y' retains its topology as a subspace of Y. Define  $\pi: Y \to X$  by  $\pi(y) = \pi'(y)$  if  $y \in Y'$  and  $\pi(p_{av}) = a$ .

Next, we make Y a Riemann surface. Add to the charts of the complex structure of Y' the following charts. Let  $V_{av} = V_{av}^* \cup \{p_{av}\}$  and let  $\zeta_{av}: V_{av} \to D$  be the continuation of the aforementioned  $\zeta_{av}$  obtained by setting  $\zeta_{av}(p_{av}) = 0$ . These charts are holomorphically compatible and everything works out nicely.

**THEOREM 2.5.** Let  $\pi: Y \to X$  and  $\tau: Z \to X$  be proper holomorphic covering maps. Let  $A \subseteq X$  be a closed discrete set and  $X' = X \setminus A$ ,  $Y' = \pi^{-1}(X')$  and  $Z' = \tau^{-1}(X')$ . Then every biholomorphic mapping  $\sigma': Y' \to Z'$  making



commute can be extended to a biholomorphic mapping  $\sigma: Y \to Z$  making



commute. In particular,  $Deck(Y/X) \cong Deck(Y'/X')$  via this extension.

*Proof.* Suppose  $a \in A$  and (U, z) is a coordinate neighborhood of a such that z(a) = 0 and z(U) is the unit disk. Let  $U^* = U \setminus \{a\}$ . We may also assume that U is so small that  $\pi$  and  $\tau$  are unbranched over  $U^*$ . Let  $V_1, \ldots, V_n$  (resp.  $W_1, \ldots, W_m$ ) be the connected components of  $\pi^{-1}(U)$  (resp.  $\tau^{-1}(U)$ ). Then  $V_v^* = V_v \setminus \pi^{-1}(a)$  (resp.  $W_\mu^*$ ) are the connected components of  $\pi^{-1}(U^*)$  (resp.  $\tau^{-1}(U^*)$ ).

Since  $\sigma':\pi^{-1}(U^*)\to \tau^{-1}(U^*)$  is biholomorphic, n=m and one may renumber so that  $\sigma'(V_v^*)=W_v^*$ . The restriction  $\pi:V_v^*\to U^*$  is a finite sheeted unbranched covering of something biholomorphic to the punctured unit disk. It follows from Theorem 1.25 that  $V_v\cap\pi^{-1}(a)$  (resp.  $W_v\cap\tau^{-1}(a)$ ) consists of only one point  $b_v$  (resp.  $c_v$ ). Hence,  $\sigma':\pi^{-1}(U^*)\to\tau^{-1}(U^*)$  can be continued to a bijection  $\pi^{-1}(U)\to\tau^{-1}(U)$ . This continuation is a homeomorphism. Also recall that the  $V_v$  and  $W_v$ 's are biholomorphic to the unit disk and hence, by Riemann's Removable Singularities Theorem, this extension is biholomorphic. If one applies this construction to every exceptional point  $a\in A$ , then one gets the desired continuation  $\sigma:Y\to Z$ .

Note that there is a canonical restriction map  $Deck(Y/X) \to Deck(Y'/X')$  which is surjective because of what we have proved above. The injectivity is a trivial consequence of the identity theorem.

## §3 THE INVERSE GALOIS PROBLEM OVER $\mathbb{C}(t)$

**THEOREM 3.1 (RIEMANN EXISTENCE THEOREM).** Meromorphic functions on a compact Riemann surface separate points.

**THEOREM 3.2.** Every finite group can be realised as the Galois group of a field extension of  $\mathbb{C}(t)$ .

*Proof.* Let *G* be a finite group having *n* elements. There is a surjection  $\mathfrak{F}_n \to G$ , where  $\mathfrak{F}_n$  is the free group on *n* elements. Recall that  $\pi_1\left(\mathbb{P}^1\setminus\{x_0,\ldots,x_{n+1}\}\right)\cong\mathfrak{F}_n$  whence due to the Galois theory of covering spaces for manifolds, there is a topological *n*-sheeted covering  $\pi: Y' \to \mathbb{P}^1\setminus\{x_0,\ldots,x_{n+1}\}$ . Note that this covering endows Y' with a unique Riemann surface structure. Since the covering has finitely many sheets, Y is compact. Due to Theorem 2.4,  $\pi$  can be extended to a branched covering  $\pi: Y \to \mathbb{P}^1$ .

For any  $\sigma \in \operatorname{Deck}(Y/\mathbb{P}^1)$ , the induced map  $\sigma^*$  on  $\mathscr{M}(Y)$  is an element of  $\operatorname{Aut}(\mathscr{M}(Y)/\mathscr{M}(\mathbb{P}^1))$ . This gives a natural group homomorphism:

$$\operatorname{Deck}(Y/\mathbb{P}^1) \longrightarrow \operatorname{Aut}(\mathscr{M}(Y)/\mathscr{M}(\mathbb{P}^1)), \quad \sigma \mapsto \sigma^*.$$

We contend that this map is injective. Indeed, suppose  $\sigma^*$  is the identity map for some  $\sigma \neq 1$ . This is equivalent to stating that  $f = f \circ \sigma$  for every  $f \in \mathcal{M}(Y)$ , which is impossible due to Theorem 3.1.

Due to Theorem 2.5, the cardinality of  $\operatorname{Deck}(Y/\mathbb{P}^1)$  is precisely the cardinality of  $\operatorname{Deck}(Y'/\mathbb{P}^1 \setminus \{x_0, \dots, x_{n+1}\})$ , which is equal to n. Further, using Theorem 3.1 and Theorem 2.3, note that  $[\mathscr{M}(Y) : \mathscr{M}(\mathbb{P}^1)] = n$ . Injectivity of the aforementioned map forces the cardinality of  $\operatorname{Aut}(\mathscr{M}(Y)/\mathscr{M}(\mathbb{P}^1))$  to be n whence the extension is Galois and the map is an isomorphism. This gives  $\operatorname{Aut}(\mathscr{M}(Y)/\mathscr{M}(\mathbb{P}^1)) \cong G$ , thereby completing the proof.