The Inverse Galois Problem over $\mathbb{C}(t)$

Swayam Chube

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Abstract

This is an attempt to present a self-contained proof of the Inverse Galois Problem over $\mathbb{C}(t)$. The only result used without proof is Riemann's Existence Theorem (Theorem 3.1).

§1 Riemann Surfaces and Holomorphic Maps

Definition 1.1. Let X be a two-dimensional manifold. A *complex chart* on X is a homeomorphism $\varphi : U \to V$ of an open subset $U \subseteq X$ onto an open subset $V \subseteq \mathbb{C}$.

Two complex charts $\varphi_i : U_i \to V_i$, $i = 1, 2$ are said to be *holomorphically compatible* if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is biholomorphic.

A *complex atlas* on X is a system $\mathfrak{A} = \{\varphi_i : U_i \to V_i \mid i \in I\}$ of charts which are holomorhpically compatible and which cover X. Two complex atlases $\mathfrak{A}$ and $\mathfrak{A}'$ on X are said to be *analytically equivalent* if every chart of $\mathfrak{A}$ is holomorphically compatible with every chart of $\mathfrak{A}'$.

A *complex structure* on a two-dimensional manifold X is an equivalence class of analytically equivalent atlases on X.

A *Riemann surface* is a pair (X, Σ) where X is a connected two-dimensional manifold and Σ is a complex structure on X.

Example 1.2. Consider $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$, the one-point compactification of $\mathbb{C}$. Let $U_1 = \mathbb{C} \subseteq \mathbb{P}^1$ and $U_2 = \mathbb{C}^* \cup \{\infty\}$. Consider the charts $\varphi_1 : U_1 \to \mathbb{C}$, the identity map, and $\varphi_2 : U_2 \to \mathbb{C}$ given by

$$\varphi(z) = \begin{cases} \frac{1}{z} & z \in \mathbb{C}^* \\ 0 & z = \infty. \end{cases}$$

These are compatible charts since the transition function is $z \mapsto \frac{1}{z}$ on $\mathbb{C}^*$.

Example 1.3. If X is a Riemann surface and $Y \subseteq X$ is a connected open set, then every chart of X restricts to a chart on Y (by restriction of the domain) and these are still holomorphically comptible. Thus, Y inherits a natural Riemann surface structure from X. In particular, every open subset of $\mathbb{C}$ is a Riemann surface.

DEFINITION 1.4. A map $f : X \to Y$ of Riemann surfaces is said to be *holomorphic* if for every pair of charts $\psi_1 : U_1 \to V_1$ on X and $\psi_2 : U_2 \to V_2$ on Y with $f(U_1) \subseteq U_2$, the mapping $\psi_2 \circ f \circ \psi_1^{-1} : V_1 \to V_2$ is holomorphic.

A holomorphic function on X means a holomorphic function $f : X \to \mathbb{C}$. These form a ring denoted by $\mathscr{O}(X)$.

THEOREM 1.5 (RIEMANN'S REMOVABLE SINGULARITIES THEOREM). Let X be a Riemann surface and $f \in \mathscr{O}(X \setminus \{a\})$. If f is bounded in a neighborhood of a, then f can be extended to a holomorphic function $\widetilde{f} \in \mathscr{O}(X)$.

Proof. Follows from the analogous statement in elementary complex analysis. ■

THEOREM 1.6 (IDENTITY THEOREM). Let X and Y be Riemann surfaces and $f_1, f_2 : X \to Y$ be two holomorphic mappings which coincide on a set $A \subseteq X$ having a limit point $a \in X$. Then $f_1 = f_2$.

Proof. Let

$$B = \{x \in X \colon \text{there is a neighborhood } W \text{ of } x \text{ such that } f_1|_W = f_2|_W\}.$$

By definition, B is open. By continuity, note that $a \in A$. Considering charts centered at a and $f_1(a) = f_2(a)$, and using the identity theorem from elementary complex analysis, it is not hard to see that $a \in B$, that is, $B \neq \emptyset$. Finally, suppose $b_n \to b \in X$. Then, by continuity, $y = f_1(b) = f_2(b)$. Consider charts centered at b and y. Note that b_n lies in the chart centred at b for sufficiently large n and hence, it would follow that $b \in B$. Thus, B is a clopen nonempty subset of X. Owing to the connectedness of X, $B = X$. This completes the proof. ■

DEFINITION 1.7. A *meromorphic function* on a Riemann surface X is a holomorphic function $f : X' \to \mathbb{C}$, where $X' \subseteq X$ is an open subset, such that the following hold:

(a) $X \setminus X'$ is discrete.

(b) For every $p \in X \setminus X'$,
$$\lim_{x \to p} |f(x)| = \infty.$$

The points of $X \setminus X'$ are called the *poles* of f. The set of all meromorphic functions on X is denoted by $\mathscr{M}(X)$.

PROPOSITION 1.8. There is a canonical correspondence between $\mathscr{M}(X)$ and the set of holomorphic functions $X \to \mathbb{P}^1$.

Proof. Straightforward. ■

COROLLARY. $\mathscr{M}(X)$ is a field.

§§ Local Normal Form

THEOREM 1.9. Let X and Y be Riemann surfaces and $f : X \to Y$ a non-constant holomorphic map. Suppose $a \in X$ and $b = f(a) \in Y$. Then, there exists an integer $k \geqslant 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y with the following properties:

(i) $a \in U$, $\varphi(a) = 0$, $b \in U'$ and $\psi'(b) = 0$.

(ii) $f(U) \subseteq U'$.

(iii) The diagram

$$\begin{array}{ccc} U & \longrightarrow & U' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ V & \xrightarrow[z \mapsto z^k]{} & V' \end{array}$$

commutes. The number k is called the *multiplicity* of f at a.

Proof. Begin with two charts $\varphi_1 : U \to V_1$ and $\psi_1 : U' \to V_1'$ satisfying (i) and (ii). The induced map $V_1 \to V_1'$ takes 0 to 0 and hence, is of the form $z^k g(z)$ for some $k \geqslant 1$ and holomorphic $g : V_1 \to V_1'$ with $g(0) \neq 0$. Shrinking all the open sets if necessary, we may suppose that $g(z) = h(z)^k$ for some holomorphic function $h : V_1 \to \mathbb{C}$. Note that $zh(z)$ must be injective and non-constant on V_1 whence maps V_1 biholomorphically onto some $V \subseteq \mathbb{C}$. We obtain the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\psi_1} \\ V_1 & \xrightarrow{(zh(z))^k} & V_1' \\ {\scriptstyle zh(z)}\downarrow & \nearrow{\scriptstyle z \mapsto z^k} & \\ V & & \end{array}$$

thereby completing the proof. ∎

THEOREM 1.10 (OPEN MAPPING THEOREM). A non-constant holomorphic map between Riemann surfaces is open.

Proof. Since being open is a local property, this follows immediately from Theorem 1.9. ∎

COROLLARY. Let $f : X \to Y$ be an injective holomorphic map of Riemann surfaces. Then f is a biholomorphic mapping of X onto $Z = f(X)$.

Proof. Due to Theorem 1.10, $Z \subseteq Y$ is open. Since f is injective, it follows from Theorem 1.9 that $k = 1$ at each point of X. In particular, f is a local homeomorphism onto Z. The conclusion follows. ∎

THEOREM 1.11. If X is a compact Riemann surface and $f : X \to Y$ a non-constant holomorphic map of Riemann surfaces, then f is surjective.

Proof. The image of f is both open and closed in Y. ∎

COROLLARY. If X is a compact Riemann surface, then $\mathscr{O}(X)$ consists of only constant functions.

§§ Branched and Unbranched Coverings

Definition 1.12. Let $p : Y \to X$ be a non-constant holomorphic map of Riemann surfaces. A point $y \in Y$ is said to be a *branch point* or *ramification point* of p, if there is no neighborhood V of y such that $p|_V$ is injective. The map p is called an *unbranched holomorphic map* if it has no branch points.

Theorem 1.13. A non-constant holomorphic map $p : Y \to X$ is unbranched if and only if p is a local homeomorphism, i.e., every point $y \in Y$ has an open neighborhood V which is mapped homeomorphically by p onto an open set U in X.

Proof. Immediate from the definition since an injective map of Riemann surfaces is a biholomorphism onto its image. ∎

Theorem 1.14. Let X be a Riemann surface, Y a connected Hausdorff topological space, and $p : Y \to X$ a local homeomorphism. Then there is a unique complex structure on Y such that p is holomorphic.

Proof. Suppose $\varphi_1 : U_1 \to V_1 \subseteq \mathbb{C}$ is a chart of the complex structure of X such that there is an open subset $U \subseteq Y$ with $p|_U : U \to U_1$ a homeomorphism. Then, $\varphi := \varphi \circ p : U \to V$ is a complex chart on Y. Let $\mathfrak{A}$ be the set of all complex charts on Y obtained in this way. It is easy to see that the charts of $\mathfrak{A}$ cover Y and are holomorphically compatible. Thus, we have defined a complex structure on Y and it follows that p is a holomorphic map when Y is equipped with this structure.

Suppose (Y, Σ) and (Y, Σ') are two complex charts such that p is holomorphic, then $\mathbf{id} : (Y, \Sigma) \to (Y, \Sigma')$ is a bijective holomorphic map, whence a biholomorphism. This shows uniqueness. ∎

Theorem 1.15. Let X, Y, Z be Riemann surfaces, $p : Y \to X$ an unbranched holomorphic map and $f : Z \to X$ any holomorphic map. Then, every continuous lift $g : Z \to Y$ of f is holomorphic.

Proof. Let $z \in Z$, $x = f(z)$, and $y = g(z)$. There is a neighborhood V of y in Y such that $p|_V$ is injective. Let $U = p(V) \subseteq X$, which is open and biholomorphic to V through p. If $W = g^{-1}(V)$, then $g|_W = p|_V^{-1} \circ f|_W$ whence g is holomorphic. ∎

Definition 1.16. A continuous map $f : X \to Y$ of topological spaces is said to be *proper* if $f^{-1}(K)$ is compact in X for every compact subset K of Y. The map f is said to be *discrete* if every fiber is discrete in X.

Lemma 1.17. A proper map between locally compact Hausdorff spaces is closed.

Proof. Follows from the fact that a subset of an LCH space is closed if and only if its intersection with every compact subset is closed. ∎

Corollary. A proper holomorphic map between Riemann surfaces is surjective.

Proof. The image is both closed and open. ∎

LEMMA 1.18. Let X and Y be locally compact Hausdorff. If $p : Y \to X$ is a proper, discrete map then:

(a) for every $x \in X$, the set $p^{-1}(x)$ is finite.

(b) if $x \in X$ and V is a neighborhood of $p^{-1}(x)$, then there is a neighborhood U of x with $p^{-1}(U) \subseteq V$.

Proof. (a) Compact discrete sets must be finite.

(b) Since $Y \setminus V$ is closed, due to the preceding lemma, $A = p(Y \setminus V)$ is closed in X and $x \notin A$. Hence, $U = X \setminus A$ is an open neighborhood of x such that $p^{-1}(U) \subseteq V$. ∎

THEOREM 1.19. Let X and Y be locally compact Hausdorff spaces and $p : Y \to X$ a proper local homeomorphism. Then p is a covering map.

Proof. Choose any $x \in X$ and let $p^{-1}(x) = \{y_1, \ldots, y_n\}$. Since p is a local homeomorphism, we can inductively choose disjoint neighborhoods W_i of y_i and a neighborhood V of x such that the restriction $p|_{W_i} : W_i \to V$ is a homeomorphism. It follows that p is a covering map. ∎

PROPOSITION 1.20. The set of branch points of a non-constant holomorphic map between Riemann surfaces is a discrete closed set.

Proof. Let $f : X \to Y$ be a non-constant holomorphic map. Let $a \in X$ be a branch point and $b = f(a)$. Then due to Theorem 1.9, there are charts $\varphi : U \to V$ and $\varphi' : U' \to V'$ centered at a and b respectively such that the induced map $V \to V'$ is $z \mapsto z^k$ for some positive integer $k \geqslant 2$ (since a is a branch point). But for any $0 \neq z \in V'$, the map $V \to V'$ is a local homeomorphism and hence, the set of branch points forms a discrete set.

To see that it is closed, let $a \in X$ not be a branch point. Then, there is a neighborhood V of a on which f is injective and hence, none of the points in V are branch points. This shows that the set of branch points is also closed. ∎

DEFINITION 1.21. Let $f : X \to Y$ be a proper holomorphic map. As we have seen earlier, f is surjective. Let $A \subseteq X$ be the set of branch points of f. Since f is proper, the set $B = f(A) \subseteq Y$ is closed and discrete (use the Local Normal Form). One calls B the set of *critical values* of f.

With notation as above, let $Y' = Y \setminus B$ and $X' = f^{-1}(Y') \subseteq X \setminus A$. The restriction $f : X' \to Y'$ is a proper unbranched holomorphic covering map since it is a local homeomorphism (owing to the fact that all branch points have been removed). It has a well-defined finite number of sheets, say n. Thus, every value $c \in Y'$ is taken precisely n times. We would like to extend this notion to critical values.

For $x \in X$, denote by $V(f, x)$, the multiplicity of f at x in the sense of Theorem 1.9. We say that f takes the value $c \in Y$, counting multiplicities, m times on X, if

$$m = \sum_{x \in f^{-1}(c)} v(f, x).$$

THEOREM 1.22. Let $f : X \to Y$ be a proper non-constant holomorphic map between Riemann surfaces. Then there exists a natural number n such that f atkes every value $c \in Y$, counting multiplicities, n times.

Proof. Using the notation as in the preceding paragraph, let n be the number of sheets of the unbranched covering $f : X' \to Y'$. Suppose $b \in B$ is a critical value, $p^{-1}(b) = \{x_1, \dots, x_r\}$ and $k_i = v(f, x_i)$. Due to Theorem 1.9, there are disjoint neighborhoods U_j of x_j and V_j of b such that for every $c \in V_j \setminus \{b\}$ the set $p^{-1}(c) \cap U_j$ consists of exactly k_j points. Due to Lemma 1.18, we can find a neighborhood $V \subseteq V_1 \cap \cdots \cap V_r$ of b such that $p^{-1}(V) \subseteq U_1 \cup \cdots \cup U_r$. Then for every point $c \in V \cap Y'$, we have that $p^{-1}(c)$ consists of $k_1 + \cdots + k_r$ points. On the other hand, the cardinality of $p^{-1}(c)$ must be the number of sheets, n and hence, $n = k_1 + \cdots + k_r$, thereby completing the proof. ■

REMARK 1.23. A proper non-constant holomorphic map between Riemann surfaces will be called an *n-sheeted holomorphic covering map*, where n is the integer found in the above result. Note that holomorphic covering maps are allowed to have branch points.

Let D denote the unit disk in $\mathbb{C}$ and $D^* = D \setminus \{0\}$.

THEOREM 1.24. Let $f : X \to D^*$ be an unbranched holomorphic covering map. Then one of the following holds:

(a) If the covering has an infinite number of sheets, then there exists a biholomorphic mapping $\varphi : X \to H$ of X onto the left half plane such that

$$\begin{array}{ccc} X & \xrightarrow[\sim]{\varphi} & H \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \exp} & \\ & D^* & \end{array}$$

commutes.

(b) If the covering is k-sheeted with $k < \infty$, then there exists a biholomorphic mapping $\varphi : X \to D^*$ such that

$$\begin{array}{ccc} X & \xrightarrow[\sim]{\varphi} & D^* \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle z \mapsto z^k} & \\ & D^* & \end{array}$$

commutes.

Proof. Follows from the Galois theory of covers and the fact that H is the universal cover of D^* and

$$\mathrm{Deck}(H/D^*) = \{\tau_n : n \in \mathbb{Z}\},$$

where $\tau_n(z) = z + 2n\pi i$. ■

THEOREM 1.25. Let $f : X \to D$ be a proper non-constant holomorphic map which is unbranched over $D^* = D \setminus \{0\}$. Then there is a natural number $k \geqslant 1$ and a biholomorphic map $\varphi : X \to D$ such that

$$\begin{array}{ccc} X & \xrightarrow[\sim]{\varphi} & D \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle z\mapsto z^k} & \\ & D & \end{array}$$

Proof. The preceding theorem furnishes a $k \geqslant 1$ making

$$\begin{array}{ccc} X & \xrightarrow[\sim]{\varphi} & D^* \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle z\mapsto z^k} & \\ & D^* & \end{array}$$

commute. Let $p_k : D \to D$ denote the map $z \mapsto z^k$. If we show that $f^{-1}(0)$ is a singleton, then we would be done since we could extend $\varphi : X \to D$ making the required diagram commute.

Suppose $f^{-1}(0)$ consists of n points $b_1, \ldots, b_n$, where $n \geqslant 1$. Then due to Lemma 1.18 there are disjoint open neighborhoods V_i of b_i and a disk $D(r) = \{z \in \mathbb{C} \colon |z| < r\}$, $0 < r \leqslant 1$ such that

$$f^{-1}(D(r)) \subseteq V_1 \cup \cdots \cup V_n.$$

Let $D^*(r) = D(r) \setminus \{0\}$. Since $f^{-1}(D^*(r))$ is homeomorphic to $p_k^{-1}(D^*(r)) = D^*(\sqrt[k]{r})$, it is connected. Since every point b_i is in the closure of $f^{-1}(D^*(r))$, $f^{-1}(D(r))$ is also connected. Hence, $n = 1$. This completes the proof. ■

§2 ALGEBRAIC FUNCTIONS

DEFINITION 2.1. Let $\pi : Y \to X$ be an n-sheeted *unbranched* holomorphic covering of Riemann surfaces and $f \in \mathscr{M}(Y)$. Every point $x \in X$ has an open neighborhood U such that $\pi^{-1}(U)$ is the disjoint union of open sets $V_1, \ldots, V_n$ and $\pi : V_v \to U$ is biholomorphic for $v = 1, \ldots, n$. Let $\tau_v : U \to V_v$ denote the inverse of the restricted map $\pi : V_v \to U$ and let $f_v = \tau_v^* f := f \circ \tau_v \in \mathscr{M}(U)$.

Define the *elementary symmetric functions* $c_1, \ldots, c_n \in \mathscr{M}(U)$ as

$$c_v = (-1)^v \sigma_v (f_1, \ldots, f_n),$$

where σ_v is the v-th elementary symmetric polynomial in n indeterminates.

This same construction can be carried out about every point in X and it is hard to not see that the the elementary symmetric functions glue to global meromorphic functions in $\mathscr{M}(X)$. These are known as the *elementary symmetric functions corresponding to f*.

THEOREM 2.2. Let $\pi : Y \to X$ be an n-sheeted branched holomorphic covering map. Suppose $A \subseteq X$ is a closed discrete subset containing all the critical values of π and let $B = \pi^{-1}(A)$. Suppose f is a holomorphic (resp. meromorphic) function on $Y \setminus B$ and

$c_1, \ldots, c_n \in \mathscr{O}(X \setminus A)$ (resp. $\in \mathscr{M}(X \setminus A)$) are the elementary symmetric functions of f. Then f may be continued holomorphically (resp. meromorphically) to Y precisely if all the c_v may be continued holomorphically (resp. meromorphically) to X.

Proof. Suppose $a \in A$ and $b_1, \ldots, b_m$ are the preimages of a. Suppose (U, z) is a relatively compact coordinate neighborhood centered at a and $U \cap A = \{a\}$. Note that $V \subseteq \overline{V} \subseteq \pi^{-1}(\overline{U})$, which is compact since π is proper. It follows that V is relatively compact and contains all the b_μ's.

Case 1. Suppose $f \in \mathscr{O}(Y \setminus B)$

(a) Suppose f can be continued holomorphically to all the points b_μ. Then f is bounded on V and hence, on $V \setminus \{b_1, \ldots, b_m\}$. This implies that all the c_v's are bounded on $U \setminus \{a\}$. Thus by Riemann's theorem on removable singularities, they may all be continued holomorphically to a.

(b) Suppose all the c_v can be continued holomorphically to a; then they are all bounded on $U \setminus \{a\}$. Note that for any $y \in V \setminus \{b_1, \ldots, b_m\}$, if $x = \pi(y)$, then $f(y)$ is a root of the polynomial

$$T^n + c_1(x)T^{n-1} + \cdots + c_n(x),$$

whose coefficients are uniformly bounded, whence f is bounded in a neighborhood of every b_μ and hence, can be continued there.

Case 2. Now suppose $f \in \mathscr{M}(Y \setminus B)$.

(a) Assume first that f can be continued meromorphically to all points b_μ. The function $\varphi = \pi^* z = z \circ \pi \in \mathscr{O}(V)$ vanishes at all the points b_μ. Thus, $\varphi^k f$ may be continued holomorphically to all the points b_μ if k is sufficiently large. The elementary symmetric functions of $\varphi^k f$ are $z^{kv} c_v$ and by the first part of the proof, they may be continued holomorphically to a. Thus, all the c_v may be continued meromorphically to a.

(b) Suppose now that all the c_v can be continued meromorphically to a. There is a sufficiently large k such that all the $z^{kv} c_v$ can be continued holomorphically to a. Thus due to the first case, $\varphi^k f$ admits a holomorphic continuation to all the points b_μ. This completes the proof. ∎

THEOREM 2.3. Let $\pi : Y \to X$ be a branched holomorphic n-sheeted covering map. If $f \in \mathscr{M}(Y)$ and $c_1, \ldots, c_n \in \mathscr{M}(X)$ are the elementary symmetric functions of f, then

$$f^n + (\pi^* c_1) f^{n-1} + \cdots + (\pi^* c_{n-1}) f + \pi^* c_n = 0.$$

- The morphism $\pi^* : \mathscr{M}(X) \hookrightarrow \mathscr{M}(Y)$ is an algebraic field extension of degree $\leqslant n$.
- Moreover, if there exists an $f \in \mathscr{M}(X)$ and an $x \in X$ with preimages $y_1, \ldots, y_n \in Y$ such that the values $f(y_v)$ for $v = 1, \ldots, n$ are all distinct, then the field extension $\pi^* : \mathscr{M}(X) \hookrightarrow \mathscr{M}(Y)$ has degree n.

Proof. The fact that f solves the equation follows immediately from the definition of the elementary symmetric functions. Let $L = \mathscr{M}(Y)$ and $K = \mathscr{M}(X)$. Choose $f_0 \in L$ maximizing $n_0 = [K(f_0) : K] \leqslant n$. Let $f \in L$ be arbitrary. Then, $K(f_0, f)$ is a finite extension of K and hence, is of the form $K(g_0)$ due to the Primitive Element Theorem. But then

$$n_0 \geqslant [K(g_0) : K] = [K(f_0, f) : K] \geqslant [K(f_0) : K] = n_0,$$

whence $f \in K(f_0)$, that is, $K(f_0) = L$ and hence, $[L : K] = n_0 \leqslant n$.

Now, consider f as in the second part of the theorem and suppose its minimal polynomial over K looks like

$$f^m + (\pi^* d_1) f^{m-1} + \cdots + (\pi^* d_m) = 0,$$

where $d_1, \ldots, d_m \in K$. Under π, $y_1, \ldots, y_n$ map to x and hence,

$$f(y_i)^m + d_1(x) f(y_i)^{m-1} + \cdots + d_m(x) = 0,$$

but since the $f(y_i)$'s are distinct, we must have $m \geqslant n$, and hence, $m = n$. This completes the proof. ∎

THEOREM 2.4. Suppose X is a Riemann surface, $A \subseteq X$ is a closed discrete subset and let $X' = X \setminus A$. Suppose Y' is another Riemann surface and $\pi' : Y' \to X'$ a proper *unbranched* holomorphic covering. Then π' extends to a branched covering of X, i.e., there exists a Riemann surface Y, a proper holomorphic mapping $\pi : Y \to X$ and a biholomorphic mapping $\varphi : Y \setminus \pi^{-1}(A) \to Y'$ making the diagram

$$\begin{array}{ccc} Y \setminus \pi^{-1}(A) & \xrightarrow[\sim]{\ \varphi\ } & Y' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X \setminus A & \end{array}$$

Proof. For every $a \in A$, choose a coordinate neighborhood (U_a, z_a) on X such that $z_a(a) = 0$, $z_a(U_a)$ is the unit disk in $\mathbb{C}$ and $U_a \cap U'_a = \emptyset$ if $a \neq a'$. Let $U_a^* = U_a \setminus \{a\}$. Since $\pi' : Y' \to X'$ is proper, $\pi'^{-1}(U_a^*)$ consists of a finite number of connected components V_{av}^*, $v = 1, \ldots, n(a)$.

For every v, the restricted mapping $\pi' : V_{av}^* \to U_a^*$ is an unbranched covering. Let its covering number be k_{av}. Due to Theorem 1.24 there are biholomorphic maps $\zeta_{av} : V_{av}^* \to D^*$ such that

$$\begin{CD} V_{av}^* @>{\zeta_{av}}>> D^* \\ @V{\pi'}VV @VV{\pi_{av}}V \\ U_a^* @>>{z_a}> D^* \end{CD}$$

commutes.

Next, let p_{av} for $a \in A$ and $v = 1, \ldots, n(a)$ be fresh points disjoint from Y' and set

$$Y = Y' \cup \{p_{av} \colon a \in A,\ v = 1, \ldots, n(a)\}.$$

We now topologize Y. If W_i, $i \in I$ is a neighborhood basis of a, then $\{p_{av}\} \cup \left(\pi'^{-1}(W_i) \cap V_{av}^*\right)$, $i \in I$ is set as a neighborhood basis of p_{av} along with the fact that Y' retains its topology as a subspace of Y. Define $\pi : Y \to X$ by $\pi(y) = \pi'(y)$ if $y \in Y'$ and $\pi(p_{av}) = a$.

Next, we make Y a Riemann surface. Add to the charts of the complex structure of Y' the following charts. Let $V_{av} = V_{av}^* \cup \{p_{av}\}$ and let $\zeta_{av} : V_{av} \to D$ be the continuation of the aforementioned ζ_{av} obtained by setting $\zeta_{av}(p_{av}) = 0$. These charts are holomorphically compatible and everything works out nicely. ∎

THEOREM 2.5. Let $\pi : Y \to X$ and $\tau : Z \to X$ be proper holomorphic covering maps. Let $A \subseteq X$ be a closed discrete set and $X' = X \setminus A$, $Y' = \pi^{-1}(X')$ and $Z' = \tau^{-1}(X')$. Then every biholomorphic mapping $\sigma' : Y' \to Z'$ making

$$\begin{array}{ccc} Y' & \xrightarrow{\sigma'} & Z' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & X' & \end{array}$$

commute can be extended to a biholomorphic mapping $\sigma : Y \to Z$ making

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Z \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & X & \end{array}$$

commute. In particular, $\mathrm{Deck}(Y/X) \cong \mathrm{Deck}(Y'/X')$ via this extension.

Proof. Suppose $a \in A$ and (U, z) is a coordinate neighborhood of a such that $z(a) = 0$ and $z(U)$ is the unit disk. Let $U^* = U \setminus \{a\}$. We may also assume that U is so small that π and τ are unbranched over U^*. Let $V_1, \ldots, V_n$ (resp. $W_1, \ldots, W_m$) be the connected components of $\pi^{-1}(U)$ (resp. $\tau^{-1}(U)$). Then $V_v^* = V_v \setminus \pi^{-1}(a)$ (resp. W_μ^*) are the connected components of $\pi^{-1}(U^*)$ (resp. $\tau^{-1}(U^*)$).

Since $\sigma' : \pi^{-1}(U^*) \to \tau^{-1}(U^*)$ is biholomorphic, $n = m$ and one may renumber so that $\sigma'(V_v^*) = W_v^*$. The restriction $\pi : V_v^* \to U^*$ is a finite sheeted unbranched covering of something biholomorphic to the punctured unit disk. It follows from Theorem 1.25 that $V_v \cap \pi^{-1}(a)$ (resp. $W_v \cap \tau^{-1}(a)$) consists of only one point b_v (resp. c_v). Hence, $\sigma' : \pi^{-1}(U^*) \to \tau^{-1}(U^*)$ can be continued to a bijection $\pi^{-1}(U) \to \tau^{-1}(U)$. This continuation is a homeomorphism. Also recall that the V_v and W_v's are biholomorphic to the unit disk and hence, by Riemann's Removable Singularities Theorem, this extension is biholomorphic. If one applies this construction to every exceptional point $a \in A$, then one gets the desired continuation $\sigma : Y \to Z$.

Note that there is a canonical restriction map $\mathrm{Deck}(Y/X) \to \mathrm{Deck}(Y'/X')$ which is surjective because of what we have proved above. The injectivity is a trivial consequence of the identity theorem. ∎

§3 THE INVERSE GALOIS PROBLEM OVER $\mathbb{C}(t)$

THEOREM 3.1 (RIEMANN EXISTENCE THEOREM). Meromorphic functions on a compact Riemann surface separate points.

THEOREM 3.2. Every finite group can be realised as the Galois group of a field extension of $\mathbb{C}(t)$.

Proof. Let G be a finite group having n elements. There is a surjection $\mathfrak{F}_n \twoheadrightarrow G$, where $\mathfrak{F}_n$ is the free group on n elements. Recall that $\pi_1\left(\mathbb{P}^1 \setminus \{x_0, \ldots, x_{n+1}\right) \cong \mathfrak{F}_n$ whence due to the Galois theory of covering spaces for manifolds, there is a topological n-sheeted covering $\pi : Y' \to \mathbb{P}^1 \setminus \{x_0, \ldots, x_{n+1}\}$. Note that this covering endows Y' with a unique Riemann surface structure. Since the covering has finitely many sheets, Y is compact. Due to Theorem 2.4, π can be extended to a branched covering $\pi : Y \to \mathbb{P}^1$.

For any $\sigma \in \mathrm{Deck}(Y/\mathbb{P}^1)$, the induced map σ^* on $\mathscr{M}(Y)$ is an element of $\mathrm{Aut}(\mathscr{M}(Y)/\mathscr{M}(\mathbb{P}^1))$. This gives a natural group homomorphism:

$$\mathrm{Deck}(Y/\mathbb{P}^1) \longrightarrow \mathrm{Aut}(\mathscr{M}(Y)/\mathscr{M}(\mathbb{P}^1)), \quad \sigma \mapsto \sigma^*.$$

We contend that this map is injective. Indeed, suppose σ^* is the identity map for some $\sigma \neq 1$. This is equivalent to stating that $f = f \circ \sigma$ for every $f \in \mathscr{M}(Y)$, which is impossible due to Theorem 3.1.

Due to Theorem 2.5, the cardinality of $\mathrm{Deck}(Y/\mathbb{P}^1)$ is precisely the cardinality of $\mathrm{Deck}(Y'/\mathbb{P}^1 \setminus \{x_0, \ldots, x_{n+1}\})$, which is equal to n. Further, using Theorem 3.1 and Theorem 2.3, note that $[\mathscr{M}(Y) : \mathscr{M}(\mathbb{P}^1)] = n$. Injectivity of the aforementioned map forces the cardinality of $\mathrm{Aut}(\mathscr{M}(Y)/\mathscr{M}(\mathbb{P}^1))$ to be n whence the extension is Galois and the map is an isomorphism. This gives $\mathrm{Aut}(\mathscr{M}(Y)/\mathscr{M}(\mathbb{P}^1)) \cong G$, thereby completing the proof. ■