Projective, Injective, and Flat Modules

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§1 Projective Modules

Definition 1.1. An A-module M is said to be *projective* if the functor $\operatorname{Hom}_A(M, -) : \mathfrak{Mod}_A \to \mathfrak{Mod}_A$ is exact.

§§ Kaplansky's Theorem

Theorem 1.2. Let $(A, \mathfrak{m}, k)$ be a local ring. If M is a projective A-module, then M is free.

We begin by proving two lemmas.

Lemma 1.3. Let R be any (commutative) ring, and F an A-module which is a direct sum of countably generated submodules. If M is a direct summand of F, then M is also a direct sum of countably generated submodules.

Proof. Let $F = M \oplus N$ and $F = \bigoplus_{\lambda \in \Lambda} E_\lambda$ where each E_λ is a countably generated R-submodule of F. Our first order of business will be to construct, using transfinite induction, a sequence of submodules $(F_\alpha)_{\alpha \in \mathbf{Ord}}$ of F such that

(i) if $\alpha < \beta$, then $F_\alpha \subseteq F_\beta$.

(ii) $F = \bigcup_\alpha F_\alpha$.

(iii) if α is a limit ordinal, then $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$.

(iv) $F_{\alpha+1}/F_\alpha$ is countably generated.

(v) $F_\alpha = M_\alpha \oplus N_\alpha$, where $M_\alpha = F_\alpha \cap M$ and $N_\alpha = F_\alpha \cap N$.

(vi) each F_α is a direct sum of a suitable subset of $\{E_\lambda : \lambda \in \Lambda\}$.

Begin by setting $F_0 = 0$. Suppose for an ordinal $\alpha > 0$, F_β has been defined for all ordinals $\beta < \alpha$. If α is a limit ordinal then set

$$F_\alpha = \bigcup_{\beta<\alpha} F_\beta.$$

We must show that F_α satisfies the aforementioned conditions. Clearly (i) and (iii) are satisfied; and further since each F_β is a direct sum of a subset of $\{E_\lambda : \lambda \in \Lambda\}$, it would follow that so is F_α, thereby verifying (vi). To verify (v), it suffices to show that $F_\alpha = M_\alpha + N_\alpha$, but this is clear since any element of F_α is also an element of F_β for some $\beta < \alpha$.

Next, suppose α is not a limit ordinal so that $\alpha = \beta + 1$ for some ordinal β. This construction is a bit involved. First, if $F_\beta = F$, then the construction stops at β. Suppose now that $F_\beta \subsetneq F$. Let Q_1 be any one of the E_λ *not* contained in F_β. Take a countable set of generators $x_{11}, x_{12}, \dots$ of Q_1. Since $F = M \oplus N$, we can write

$$x_{11} = m_{11} + n_{11} \quad \text{for } m_{11} \in M \text{ and } n_{11} \in N.$$

Further, using the decomposition $F = \bigoplus_{\lambda\in\Lambda} E_\lambda$, we can write

$$m_{11} = \sum_{\substack{\lambda\in\Lambda \\ \text{finite}}} m_{11}^\lambda \quad \text{and} \quad n_{11} = \sum_{\substack{\lambda\in\Lambda \\ \text{finite}}} n_{11}^\lambda.$$

Now let Q_2 be the sum of those E_λ's for which λ occurs in the two expressions above. Since Q_2 is a finite direct sum of some E_λ's, it is countably generated. Let $x_{21}, x_{22}, \dots$ be a countable generating set of Q_2. Just as before, we can (uniquely) decompose $x_{12} = m_{12} + n_{12}$ with $m_{12} \in M$ and $n_{12} \in N$; and further decompose

$$m_{12} = \sum_{\substack{\lambda\in\Lambda \\ \text{finite}}} m_{12}^\lambda \quad \text{and} \quad n_{12} = \sum_{\substack{\lambda\in\Lambda \\ \text{finite}}} n_{12}^\lambda.$$

Again, set Q_3 to be the direct sum of those E_λ's for which λ occurs in the two expressions above, so that Q_3 is countably generated too. Pick a countable generating set $x_{31}, x_{32}, \dots$ of Q_3. Next decompose x_{21} and repeat the procedure above to obtain Q_4 and its countable generating set $x_{41}, x_{42}, \dots$. Decompose x_{13} next and repeat ad infinitum.

$$\begin{matrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

To be explicit, the order in which we decompose the x_{ij}'s is

$$x_{11},\ x_{12},\ x_{21},\ x_{13},\ x_{22},\ x_{31},\ x_{14},\ \dots.$$

Finally, set F_α to be the submodule of F generated by F_β and $\{x_{ij} : i,j \geqslant 1\}$. Clearly F_α/F_β is countably generated and $F_\beta \subseteq F_\alpha$, which verifies (i) and (iv). Since $\{x_{ni} : i \geqslant 1\}$ generates Q_n, we in fact have

$$F_\alpha = F_\beta + \sum_{n\geqslant 1} Q_n,$$

whence F_α is a direct sum of a subset of $\{E_\lambda : \lambda \in \Lambda\}$. It remains to verify (v), and to this end, it suffices to show that $F_\alpha = M_\alpha + N_\alpha$. An element of F_α can be written as

$$f_\beta + \sum_{\substack{i,j \\ \text{finite}}} a_{ij} x_{ij},$$

for some $f_\beta \in F_\beta$ and $a_{ij} \in R$. Recall that we can write

$$x_{ij} = m_{ij} + n_{ij}, \quad m_{ij} = \sum_{\substack{\lambda\in\Lambda \\ \text{finite}}} m_{ij}^\lambda, \quad \text{and} \quad n_{ij} = \sum_{\substack{\lambda\in\Lambda \\ \text{finite}}} n_{ij}^\lambda.$$

Note that each m_{ij}^λ is contained in one of the Q_n's, and hence, in F_α. Therefore m_{ij} and n_{ij} are elements of F_α, and hence, are elements of M_α and N_α respectively. Further, by the inductive hypothesis, $f_\beta = m_\beta + n_\beta$ for some $m_\beta \in M_\beta \subseteq M_\alpha$ and $n_\beta \in N_\beta \subseteq N_\alpha$, whence it follows that $F_\alpha = M_\alpha + N_\alpha$, thereby verifying (v).

Next, note that the composition

$$F_{\alpha+1} \twoheadrightarrow M_{\alpha+1} \twoheadrightarrow M_{\alpha+1}/M_\alpha$$

has kernel containing F_α and therefore, $M_{\alpha+1}/M_\alpha$ is a quotient of $F_{\alpha+1}/F_\alpha$, which is countably generated, and hence so is $M_{\alpha+1}/M_\alpha$. Next, since M_α is a direct summand of F_α, it is also a direct summand of F. Hence, M_α is a direct summand of $M_{\alpha+1}$. Thus, we can write

$$M_{\alpha+1} = M_\alpha \oplus M'_{\alpha+1},$$

where $M'_{\alpha+1}$ is countably generated. When α is a limit ordinal, set $M'_\alpha = 0$. It is now easy to see that

$$M_\alpha = \bigoplus_{\beta\leqslant\alpha} M'_\beta.$$

And since $M = \bigcup_\alpha M_\alpha$, it follow that

$$M = \bigoplus_\alpha M'_\alpha,$$

thereby completing the proof. ■

Lemma 1.4. Let M be a projective module over a local ring $(A, \mathfrak{m})$ and $x \in M$. Then there exists a direct summand of M containing x which is a free module.

Proof. We can write F as a direct summand of a free A-module $F = M \oplus N$. Choose a basis $B = \{u_i\}_{i\in I}$ such that x has the minimum possible non-zero coefficients when expressed as an A-linear combination of the u_i's. Write

$$x = a_1 u_1 + \cdots + a_n u_n$$

for some $0 \neq a_i \in A$. Note that we must have $a_i \notin \sum_{j\neq i} A a_j$ for $1 \leqslant i \leqslant n$. Indeed, if we could write

$$a_n = b_1 a_1 + \cdots + b_{n-1} a_n,$$

then

$$x = \sum_{i=1}^{n-1} a_i(u_i + b_i u_n),$$

and $\{u_1 + b_1 u_n, \ldots, u_{n-1} + b_{n-1}u_n, u_n\} \cup \{u_j : j \neq 1, \ldots, n\}$ is also a basis of F, which would contradict the minimality in the choice of B.

Set $u_i = y_i + z_i$ where $y_i \in M$ and $z_i \in N$. Since $x \in M$, we must have

$$x = a_1 y_1 + \cdots + a_n y_n.$$

We can write each y_i in coordinates as

$$y_i = \sum_{j=1}^{n} c_{ij} u_j + t_i,$$

for some $c_{ij} \in A$ and $t_i \in F$ which is a linear combination of u_k's for $k \neq 1, \ldots, n$. Thus

$$x = \sum_{i=1}^{n} a_i y_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ij} u_j + \sum_{i=1}^{n} a_i t_i.$$

By the uniqueness of coordinate representation with respect to a basis, we get

$$a_i = \sum_{j=1}^{n} a_j c_{ji} \implies \sum_{j=1}^{n} a_j \left(c_{ji} - \delta_{ji}\right) = 0$$

for $1 \leqslant i \leqslant n$. Since elements in $A \setminus \mathfrak{m}$ are invertible, we must have that $c_{ii} \in 1 + \mathfrak{m}$ for all $1 \leqslant i \leqslant n$ and $c_{ij} \in \mathfrak{m}$ for $1 \leqslant i \neq j \leqslant n$. In particular, this means the matrix $\mathbf{C} = \left(c_{ij}\right)$ is invertible since its determinant is in $1 + \mathfrak{m}$.

We claim that $\widetilde{B} = \{y_1, \ldots, y_n\} \cup \{u_i : i \neq 1, \ldots, n\}$ is a basis for F. The invertibility of $\mathbf{C}$ shows that each u_i can be written as an A-linear combination of elements in $\widetilde{B}$, and hence, the A-linear span of $\widetilde{B}$ is all of F. To see that $\widetilde{B}$ is A-linearly independent, suppose

$$0 = \sum_{i=1}^{n} f_i y_i + \sum_{\lambda \neq 1, \ldots, n} f_\lambda u_\lambda.$$

Substituting the representation of y_i in the basis B, we have

$$0 = \sum_{i=1}^{n} f_i \left(\sum_{j=1}^{n} c_{ij} u_j + t_i \right) + \sum_{\lambda \neq 1, \ldots, n} f_\lambda u_\lambda.$$

Therefore, in particular,

$$\begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix} \mathbf{C} = 0,$$

and the invertibility of $\mathbf{C}$ would mean $f_i = 0$ for $1 \leqslant i \leqslant n$; consequently,

$$\sum_{\lambda \neq 1, \ldots, n} f_\lambda u_\lambda = 0,$$

so that $f_\lambda = 0$ for all λ. Hence $\widetilde{B}$ is a basis of F. Let F_1 denote the A-submodule generated by $\{y_1, \ldots, y_n\}$. This is a free direct summand of F contained in M, and hence, is a free direct summand of M containing x. ■

Proof of Theorem 1.2. M is a direct summand of a free module, and every free module is a direct sum of countably generated submodules. Hence M itself is a direct sum of countably generated projective modules. Therfore, it is sufficient to prove the theorem assuming M is countably generated.

Let $\{\omega_1, \omega_2, \ldots\}$ be a countable generating set for M. By Lemma 1.4, there exists a free direct summand F_1 of M containing ω_1. Write $M = F_1 \oplus M_1$ and let ω_2' denote the M_1 component of ω_2. Since M_1 is projective, using Lemma 1.4, there exists a free direct summand F_2 of M_1 containing ω_2. Then $M_1 = F_2 \oplus M_2$ so that $M = F_1 \oplus F_2 \oplus M_2$. Let ω_3' denote the M_2-component of ω_3 and repeat the above process ad infinitum. That would yield $M = F_1 \oplus F_2 \oplus \cdots$, whence M is free. ■

§§ Projective Covers

DEFINITION 1.5. Let R be a ring and M an R-module. A submodule K of M is said to be *small* if for any R-submodule N of M

$$K + N = M \implies N = M.$$

We denote this by $K \ll M$.

We give two standard examples of small submodules.

(i) Let $(R, \mathfrak{m}, k)$ be a local ring, M a finite R-module, and $K = \mathfrak{m}M$. Due to Nakayama's lemma, for any R-submodule N of M, if $N + \mathfrak{m}M = M$, then $N = M$. Thus $K \ll M$.

(ii) Similarly, let $(R, \mathfrak{m}, k)$ be an Artinian local ring, M any R-module, and $K = \mathfrak{m}M$. If N is an R-submodule of M such that $N + \mathfrak{m}M = M$, then $\mathfrak{m}(M/N) = M/N$. But since $\mathfrak{m}$ is nilpotent, we must have that $M/N = 0$, so that $M = N$. Thus $K \ll M$.

DEFINITION 1.6. Let R be a ring and M an R-module. A *projective cover* of M is a pair (P, f), where P is a projective R-module and $f : P \to M$ a surjective R-linear map such that $\ker f \ll M$.

REMARK 1.7. Unlike the situation for injective hulls, projective covers need not always exist. For example, consider the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z}$, where $p > 0$ is a rational prime. Suppose $f : P \to \mathbb{Z}/p\mathbb{Z}$ is a projective cover. Since P is a projective $\mathbb{Z}$-module, it must be free. Set $K = \ker f \ll P$. Since P/K is a simple $\mathbb{Z}$-module, K is a maximal submodule of P. On the other hand, since $K \ll P$, for any proper submodule N of K, if N were not contained in K, then $K + N = P$, whence $N = P$, a contradiction. Thus, K must contain every proper submodule of P. This is absurd, since P admits quotients of the form $\mathbb{Z}/q\mathbb{Z}$ for primes $q \neq p$.

THEOREM 1.8. Every module over an Artinian ring admits a projective cover.

Proof. Let A be an Artinian ring, so that we can write $A = A_1 \times \cdots \times A_n$ as a product of Artinian local rings $(A_i, \mathfrak{m}_i, k_i)$ for $1 \leqslant i \leqslant n$. Any A-module M is of the form $M_1 \times \cdots \times M_n$ where each M_i is an A_i-module. From this reduction, it is easy to see that it suffices to prove the theorem in the Artinian local case.

Therefore, let $(A, \mathfrak{m}, k)$ be an Artinian local ring and M an A-module. Choose a k-basis $\{\overline{x}_\lambda : \lambda \in \Lambda\}$ of $M/\mathfrak{m}M$, where $x_\lambda \in M$ for all $\lambda \in \Lambda$. Let F denote the free A-module on Λ, i.e.,

$$F = \bigoplus_{\lambda \in \Lambda} A e_\lambda,$$

and let $f : F \to M$ be the unique A-linear map sending $e_\lambda \mapsto \overline{x}_\lambda$ for all $\lambda \in \Lambda$. The k-linear independence of Λ forces $\ker f = \mathfrak{m}F \ll F$ due to (ii). Using the projectivity of F as an R-module, we can lift f to a map $\widetilde{f}: F \to M$ making

$$\begin{array}{ccc} & & M \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow \\ F & \xrightarrow[f]{} & M/\mathfrak{m}M \end{array}$$

commute. Since f is surjective, we have $\mathfrak{m}M + \operatorname{im}\widetilde{f} = M$, that is, $\mathfrak{m}\left(M/\operatorname{im}\widetilde{f}\right) = M/\operatorname{im}\widetilde{f}$. Again, since $\mathfrak{m}$ is nilpotent, we have $M/\operatorname{im}\widetilde{f} = 0$, that is, $\widetilde{f}$ is surjective. Finally, since $\ker\widetilde{f} \subseteq \ker f$, it is also a small submodule of F, and hence $(F, \widetilde{f})$ is a projective cover of M. ■

Proposition 1.9. Let M be an R-module. Then

$$\bigcap \left\{\text{maximal proper submodules of } M\right\} = \sum \left\{\text{small submodules of } M\right\}.$$

This submodule is called the *radical* of M and is denoted by $\operatorname{rad}(M)$.

Proof. If $K \ll M$ and L is any maximal proper submodule of M, then K must be contained in L, else $K + L = M$ which would impoly $L = M$, a contradiction. Thus every small submodule of M is contained in every maximal proper submodule of M. Hence, the sum of all small submodules of M is contained in the intersection of all maximal proper submodules of M.

Conversely, suppose $x \in M$ is contained in the intersection of all maximal proper submodules of M. We claim that $Rx \ll M$. Indeed, if N is a proper submodule of M such that $Rx + N = M$, then M/N is a cyclic R-module, so that it admits a non-zero simple quotient. In particular, N is contained in a maximal proper submodule K of M. But $x \in K$ by assumption; and hence $Rx + N \subseteq K \subsetneq M$, a contradiction. This shows that $Rx \ll M$, thereby completing the proof. ■

Proposition 1.10. If $\{M_\lambda\}_{\lambda \in \Lambda}$ is a collection of R-module, then

$$\operatorname{rad}\left(\bigoplus_{\lambda\in\Lambda} M_\lambda\right) = \bigoplus_{\lambda\in\Lambda} \operatorname{rad}(M_\lambda).$$

Proof. Straightforward. ■

Proposition 1.11. Let $\mathfrak{R}$ denote the Jacobson radical of a ring R. Then for any R-module M, $\mathfrak{R}M \subseteq \operatorname{rad}(M)$.

Proof. If $N \subsetneq M$ is a maximal proper submodule of M, then M/N is isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. Since $\mathfrak{R} \subseteq \mathfrak{m}$, $\mathfrak{R}M \subseteq N$. Thus $\mathfrak{R}M \subseteq \operatorname{rad}(M)$. ■

Proposition 1.12. Let R be a ring and P a projective R-module. Then $\operatorname{rad}(P) = \mathfrak{R}P$, where $\mathfrak{R}$ denotes the Jacobson radical of R.

Proof. There is an R-module Q such that $F = P \oplus Q$ is a free module. In view of Proposition 1.10 and Proposition 1.11

$$\mathfrak{R}P \oplus \mathfrak{R}Q \subseteq \operatorname{rad}(P) \oplus \operatorname{rad}(Q) = \operatorname{rad}(F) = \mathfrak{R}F = \mathfrak{R}P \oplus \mathfrak{R}Q.$$

Hence $\operatorname{rad} P = \mathfrak{R}P$. ■

Theorem 1.13. A flat module over an Artinian ring is projective.

Proof. Let R be an Artinian ring and E a flat R-module. In view of Theorem 1.8, there exists a projective cover $f: P \to E$. Set $K = \ker f \ll P$, and hence $\ker f \subseteq \mathfrak{R}P$. Now since P and E are flat R-modules, the "multiplication maps"

$$\mu_1 : \mathfrak{R} \otimes_R P \to \mathfrak{R}P \quad \text{and} \quad \mu_2 : \mathfrak{R} \otimes_R E \to \mathfrak{R}E$$

are isomorphisms, which can be seen either using the equational criterion of flatness of just invoking the Tor long exact sequence. Further note that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{R} \otimes_R K & \xrightarrow{\mathbb{1} \otimes \iota} & \mathfrak{R} \otimes_R P & \xrightarrow{\mathbb{1} \otimes f} & \mathfrak{R} \otimes_R E & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \\ & & & & \mathfrak{R}P & \xrightarrow[g]{} & \mathfrak{R}E & & \end{array}$$

commutes where g is the restriction of f to $\mathfrak{R}P$. Since E is flat, the Tor long exact sequence gives that the top row is short exact. Using the fact that μ_1 and μ_2 are isomorphisms, we can write

$$K = \ker f = \ker g = \mu_1(\ker(\mathbb{1} \otimes f)) = \mu_1(\operatorname{im}(\mathbb{1} \otimes \iota)) = \mathfrak{R}K.$$

But $\mathfrak{R}$ is nilpotent in R, and hence $K = 0$, that is, $E \cong P$ is projective. ■

Corollary. An arbitrary product of projective modules over an Artinian ring is projective.

Proof. This follows immediately from Theorem 2.9 and Theorem 1.13. ■

§2 Flat Modules

Definition 2.1. An A-module M is said to be *flat* if the functor $- \otimes_A M : \mathfrak{Mod}_A \to \mathfrak{Mod}_A$ is exact.

Definition 2.2. Let M be an A-module and $\sum_{i=1}^n f_i x_i = 0$ be a relation in M for $f_i \in A$ and $x_i \in M$. We say that the relation is *trivial* if there exists an integer $m \geqslant 0$, elements $y_j \in M$ for $1 \leqslant j \leqslant m$ and $a_{ij} \in A$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that

$$x_i = \sum_{j=1}^m a_{ij} y_j \quad \forall\ 1 \leqslant i \leqslant n \quad \text{and} \quad 0 = \sum_{i=1}^n a_{ij} f_i \quad \forall\ 1 \leqslant j \leqslant m.$$

Lemma 2.3 (Equational criterion of flatness). An A-module M is flat if and only if every relation in M is trivial.

Proof. Suppose M is flat and $\sum_{i=1}^n f_i x_i = 0$ is a relation in M. Let $\mathfrak{a} = (f_1, \ldots, f_n) \subseteq A$ and consider the A-linear surjection $A^n = \bigoplus_{i=1}^n A e_i \to I$ given by $e_i \mapsto f_i$ whose kernel is $K \subseteq A^n$. That is, $0 \to K \to A^n \to \mathfrak{a} \to 0$. Since M is flat, tensoring with M preserves exactness and we have an exact sequence

$$0 \longrightarrow K \otimes_A M \longrightarrow A^n \otimes_A M \longrightarrow \mathfrak{a} \otimes_A M \longrightarrow 0.$$

Note that the natural map $\mathfrak{a} \otimes_A M \to R \otimes_A M$ is injective due to the flatness of M. Consequently, $\sum_{i=1}^n f_i \otimes x_i$ maps to 0 in $R \otimes_A M$ and hence, must be zero in $\mathfrak{a} \otimes_A M$. The exactness of the above sequence furnishes an element $\sum_{j=1}^m k_j \otimes y_j \in K \otimes_A M$ that maps to 0 in $A^n \otimes_A M$.

Each k_j can be written in the form

$$\sum_{i=1}^n a_{ij} e_i \quad \forall\ 1 \leqslant j \leqslant m,$$

and hence, the image of $\sum_{j=1}^m k_j \otimes y_j$ in $A^n \otimes_A M$ is

$$\sum_{j=1}^m \sum_{i=1}^m a_{ij} e_i \otimes y_j = \sum_{i=1}^n e_i \otimes \left(\sum_{j=1}^m a_{ij} y_j \right) = 0,$$

and the conclusion follows.

Conversely, suppose every relation in M is trivial and let $\mathfrak{a}$ be a finitely generated ideal of A. It suffices to show that $\operatorname{Tor}_1^A(A/\mathfrak{a}, M) = 0$, which is equivalent (from the Tor long exact sequence) to showing that the map $\mathfrak{a} \otimes_A M \to A \otimes_A M$ is injective.

Suppose $\sum_{i=1}^n f_i \otimes x_i \in \mathfrak{a} \otimes_A M$ maps to 0 in $A \otimes_A M$. Then, $\sum_{i=1}^n f_i x_i = 0$ in M, consequently, there is an $m \geqslant 0$, $y_j \in M$, $a_{ij} \in M$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that

$$x_i = \sum_{j=1}^m a_{ij} y_j \quad \forall\ 1 \leqslant i \leqslant n \quad \text{and} \quad 0 = \sum_{i=1}^n a_{ij} f_i \quad \forall\ 1 \leqslant j \leqslant m.$$

Consequently, in $\mathfrak{a} \otimes_A M$,

$$\sum_{i=1}^n f_i \otimes x_i = \sum_{i=1}^n f_i \otimes \left(\sum_{j=1}^m a_{ij} y_j \right) = \left(\sum_{i=1}^n a_{ij} f_i \right) \otimes y_j = 0.$$

This proves injectivity, thereby completing the proof. ■

LEMMA 2.4. Let $(A, \mathfrak{m}, k)$ be a local ring and M a flat A-module. If $x_1, \dots, x_n \in M$ are such that their images $\overline{x}_1, \dots, \overline{x}_n \in M/\mathfrak{m}M$ are linearly independent over k, then $x_1, \dots, x_n$ are linearly independent over A.

Proof. We prove this statement by induction on n. If $n = 1$, then $a \in A$ is such that $ax_1 = 0$ and $\overline{x}_1 \neq 0$. From Lemma 2.3, there are $b_1, \dots, b_m \in A$ and $y_1, \dots, y_m \in M$ such that

$$x_1 = \sum_{j=1}^m b_j y_j \quad \text{and} \quad ab_j = 0 \quad \forall\ 1 \leqslant j \leqslant m.$$

Since $x_1 \notin \mathfrak{m}M$, it follows that at least one of the b_j's must be a unit, whence $a = 0$.

Now, suppose $n > 1$ and there is a relation $\sum_{i=1}^n a_i x_i = 0$ in M. From Lemma 2.3, there is an $m \geqslant 0$, $y_j \in M$, and $b_{ij} \in A$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that

$$x_i = \sum_{j=1}^m b_{ij} y_j \quad \forall\ 1 \leqslant i \leqslant n \quad \text{and} \quad 0 = \sum_{i=1}^n b_{ij} a_i \quad \forall\ 1 \leqslant j \leqslant m.$$

Since $x_n \notin \mathfrak{m}M$, at least one of the b_{nj}'s must be a unit, whence we can write

$$a_n = \sum_{i=1}^{n-1} c_i a_i,$$

for some $c_i \in A$ for $1 \leqslant i \leqslant n-1$. Therefore, we have

$$0 = \sum_{i=1}^n a_i x_i = \sum_{i=1}^{n-1} a_i (x_i + c_i x_n).$$

Since $\overline{x}_1, \dots, \overline{x}_{n-1}$ are k-linearly independent in $M/\mathfrak{m}M$, we see that $\overline{x}_1 + \overline{c}_1 \overline{x}_n, \dots, \overline{x}_{n-1} + \overline{c}_{n-1} \overline{x}_n$ must also be k-linearly independent. Due to the induction hypothesis, $a_1 = \cdots = a_{n-1} = 0$ and hence, $a_n = 0$. This completes the proof. ■

THEOREM 2.5. Let $(A, \mathfrak{m}, k)$ be a local ring. If M is a finitely generated flat A-module, then M is free.

Proof. Let $x_1, \dots, x_n \in M$ be a minimal generating set, that is, $\overline{x}_1, \dots, \overline{x}_n$ are k-linearly independent in $M/\mathfrak{m}M$. Due to the preceding lemma, $x_1, \dots, x_n$ are linearly independent over A, and hence, M is a free A-module. ■

§§ Cartier's Theorem

THEOREM 2.6 (CARTIER). Let M be a finitely generated module over an integral domain A. If for every $\mathfrak{m} \in \text{MaxSpec}(A)$, $M_\mathfrak{m}$ is free as an $A_\mathfrak{m}$-module, then M is a projective A-module.

Proof. First show that M is a torsion-free A-module. Suppose $am = 0$ for some $0 \neq a \in A$ and $m \in M$. Let $\mathfrak{a}$ be the annihilator of m in A and $\mathfrak{m}$ a maximal ideal containing A. Note that $\frac{a}{1}\frac{m}{1} = 0$ in $M_\mathfrak{m}$, which is free over $A_\mathfrak{m}$, an integral domain, whence, is torsion free. That is, $\frac{m}{1} = 0$, whence, there is some $s \in A \setminus \mathfrak{m}$ such that $sm = 0$, which is absurd, since $\mathfrak{a} \subseteq \mathfrak{m}$. This shows that M is torsion-free.

Now, choose a set of generators $\{m_i : 1 \leqslant i \leqslant n\}$ for M over A. Let $\mathscr{P}$ be the collection of A-endomorphisms of M which are of the form

$$m \longmapsto \sum_{i=1}^{n} f_i(m) m_i,$$

where $f_1, \ldots, f_n : M \to A$ are A-module homomorphisms. Note that $\mathscr{P}$ is an A-submodule of $\text{End}_A(M)$. We shall show that $\mathbf{id}_M \in \mathscr{P}$.

Let $\mathfrak{m}$ be a maximal ideal of A. We know that $M_\mathfrak{m}$ is free as an $A_\mathfrak{m}$-module and hence, there are $A_\mathfrak{m}$-module homomorphisms $f_i : M_\mathfrak{m} \to A_\mathfrak{m}$ such that

$$m' = \sum_{i=1}^{n} f_i'(m') \frac{m_i}{1} \quad \forall m' \in M_\mathfrak{m}.$$

To see that this is possible, first consider an $A_\mathfrak{m}$-basis $\{e_i : 1 \leqslant i \leqslant N\}$ for $M_\mathfrak{m}$. We can write

$$e_i = \sum_{j=1}^{n} a_{ij} \frac{m_j}{1} \quad \forall\ 1 \leqslant i \leqslant N.$$

Further, there are $A_\mathfrak{m}$-linear maps $f_i : M_\mathfrak{m} \to A_\mathfrak{m}$ such that

$$m' = \sum_{j=1}^{N} f_j(m') e_j.$$

Set

$$f_j'(m') = \sum_{i=1}^{N} a_{ij} f_i(m') \quad \forall\ m' \in M_\mathfrak{m}.$$

Then,

$$\sum_{j=1}^{n} f_j'(m') \frac{m_j}{1} = \sum_{i=1}^{N} \sum_{j=1}^{n} a_{ij} f_i(m') \frac{m_j}{1} = \sum_{i=1}^{N} f_i(m') e_i = m'.$$

Coming back, since M is torsion-free, the canonical map $M \to M_\mathfrak{m}$ is an injective map of A-modules. Further, we can find an $s \in A \setminus \mathfrak{m}$ such that $sf_i'\left(\frac{m_j}{1}\right) \in A$ for $1 \leqslant i, j \leqslant n$.

Note that $m' \mapsto sf_i'(m')$ is $A_\mathfrak{m}$-linear as a map $M_\mathfrak{m} \to A_\mathfrak{m}$, and hence, is A-linear. The restriction of this map to $M \subseteq M_\mathfrak{m}$ takes values in A. Thus, we can identify sf_i' with an A-linear map $M \to A$. Further, for every $m \in M$, we have

$$sm = \sum_{i=1}^{n} sf_i'(m) m_i.$$

That is, $s \cdot \mathbf{id}_M \in \mathscr{P}$. Now, let $\mathfrak{a}$ be the collection of all $a \in A$ such that $a \cdot \mathbf{id}_M \in \mathscr{P}$. Then $\mathfrak{a}$ is an ideal of A. If $\mathfrak{a}$ were a proper ideal, it would be contained in a maximal ideal $\mathfrak{m}$. But from our preceding conclusion, there is some $s \in A \setminus \mathfrak{m}$ such that $s \cdot \mathbf{id}_M \in \mathscr{P}$, a contradiction. Thus, $\mathfrak{a} = A$, in particular, $\mathbf{id}_M \in \mathscr{P}$.

Finally, we show that M is projective. We have shown that there are A-linear maps $f_i : M \to A$ such that

$$m = \sum_{i=1}^{n} f_i(m) m_i \quad \forall\ m \in M.$$

Let F be the free module $\bigoplus_{i=1}^{n} Ae_i$ and let $g : F \to M$ be given by $e_i \mapsto m_i$ and $f : M \to F$ given by

$$f(m) = \sum_{i=1}^{n} f_i(m) e_i.$$

By our construction, $g \circ f = \mathbf{id}_M$, and hence M is a direct summand of F, i.e. M is projective. ■

COROLLARY. A finitely generated flat module over an integral domain is projective.

Proof. Follows from Theorem 2.6 and Theorem 2.5. ■

§§ Finitely Presented Modules and Flatness

THEOREM 2.7. Let M be a finitely presented A-module and N be any A-module. If B is a flat A-algebra, then there is a natural isomorphism

$$\mathrm{Hom}_A(M,N) \otimes_A B \cong \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B).$$

Proof. Fixing N and B, there are contravariant functors $\mathscr{F}, \mathscr{G} : \mathfrak{Mod}_A^{op} \to \mathfrak{Mod}_B$ given by

$$\mathscr{F}(M) = \mathrm{Hom}_A(M,N) \otimes_A B \qquad \mathscr{G}(M) = \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B).$$

Define the natural transformation $\lambda : \mathscr{F} \implies \mathscr{G}$ given by

$$\lambda_M(f \otimes b) = b \cdot (f \otimes \mathbf{id}_B).$$

We first show that this is natural in M. Indeed, suppose $\varphi : M' \to M$ is A-linear, we wish to show that

$$\begin{array}{ccc} \mathscr{F}(M) & \longrightarrow & \mathscr{F}(M') \\ {\scriptstyle \lambda_M}\downarrow & & \downarrow{\scriptstyle \lambda_{M'}} \\ \mathscr{G}(M) & \longrightarrow & \mathscr{G}(M') \end{array}$$

commutes. Consider $f \otimes b \in \mathscr{F}(M)$, which maps to $f \circ \varphi \otimes b \in \mathscr{F}(M')$, which maps to $b \cdot (f \circ \varphi \otimes \mathbf{id}_B) \in \mathscr{G}(M')$. On the other hand, under λ_M, $f \otimes b$ maps to $b \cdot (f \otimes \mathbf{id}_B) \in \mathscr{G}(M)$, which maps to $b \cdot (f \circ \varphi \otimes \mathbf{id}_B)$, which shows commutativity.

Next, suppose $M = A^n$ were free of finite rank. In this case, there is a sequence of isomorphisms

$$\mathrm{Hom}_A(A^n, N) \otimes_A B \cong N^n \otimes_A B \cong (N \otimes_A B)^n \cong \mathrm{Hom}_B(B^n, N \otimes_A B) \cong \mathrm{Hom}_B(A^n \otimes_A B, N \otimes_A B).$$

Under the above isomorphism, $f \otimes b$ first maps to $(f(e_1), \ldots, f(e_n))^\top \otimes b$ in $N^n \otimes_A B$. Under the second map, it goes to $(f(e_1) \otimes b, \ldots, f(e_n) \otimes b)^\top$ in $(N \otimes_A B)^n$. Under the third map it goes to the unique morpism $g : B^n \to N \otimes_A B$ that sends $e_i \mapsto f(e_i) \otimes b$.

Consider the map $b \cdot (f \otimes \mathbf{id}_B) \in \mathrm{Hom}_B(A^n \otimes_A B, N \otimes_A B)$. Under this map, $e_i \in B^n$ is the same as $e_i \otimes 1 \in A^n \otimes B$, which maps to $b \cdot (f(e_i) \otimes 1) = f(e_i) \otimes b \in N \otimes_A B$. It follows that this is the same as the aforementioned g. Thus, λ_M is an isomorphism in this case.

Finally, there is an exact sequence $A^m \to A^n \to M \to 0$ since M is finitely presented. This fits into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathscr{F}(M) & \longrightarrow & \mathscr{F}(A^n) & \longrightarrow & \mathscr{F}(A^m) \\ \| & & \downarrow \lambda & & \downarrow \lambda & & \downarrow \lambda \\ 0 & \longrightarrow & \mathscr{G}(M) & \longrightarrow & \mathscr{G}(A^n) & \longrightarrow & \mathscr{G}(A^m) \end{array}$$

where the last two λ's are isomorphisms. Due to the Five Lemma (after adding another column of zeros to the left), we see that $\lambda_M : \mathscr{F}(M) \to \mathscr{G}(M)$ must be an isomorphism, thereby completing the proof. ■

COROLLARY. Let M be a finitely presented A-module and N be any A-module. Then for every $\mathfrak{p} \in \operatorname{Spec}(A)$,

$$\operatorname{Hom}_A(M,N)_{\mathfrak{p}} \cong \operatorname{Hom}_{A_{\mathfrak{p}}}\left(M_{\mathfrak{p}}, N_{\mathfrak{p}}\right).$$

Proof. Note that the localization functor at $\mathfrak{p} \in \operatorname{Spec}(A)$ is naturally isomorphic to $- \otimes_A A_{\mathfrak{p}}$. ■

THEOREM 2.8. Let M be a finitely presented A-module. Then the following are equivalent

(a) M is projective.

(b) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(A)$.

(c) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \operatorname{MaxSpec}(A)$.

Proof. That $(a) \implies (b) \implies (c)$ is obvious. It suffices to show that $(c) \implies (a)$. To this end, we shall show that $\operatorname{Hom}_A(M,-)$ is an exact functor. We know that $\operatorname{Hom}_A(M,-)$ is left exact so let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence. Upon application of the above functor, note that we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(M,N') \longrightarrow \operatorname{Hom}_A(M,N) \longrightarrow \operatorname{Hom}_A(M,N'') \to K \to 0,$$

where K is the cokernel. Localizing the above sequence at a maximal ideal $\mathfrak{m}$ and using the exactness of localization and the preceding result, we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}},N'_{\mathfrak{m}}) \longrightarrow \operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}},N_{\mathfrak{m}}) \longrightarrow \operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}},N''_{\mathfrak{m}}) \to K_{\mathfrak{m}} \to 0.$$

But since $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module, the functor $\operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}},-)$ is exact, whence $K_{\mathfrak{m}} = 0$ for every $\mathfrak{m} \in \operatorname{MaxSpec}(A)$. This shows that $K = 0$, that is, M is projective. ■

THEOREM 2.9. Let A be a Noetherian ring and $\{M_\lambda\}_{\lambda \in \Lambda}$ a family of flat A-modules. Then $M = \prod_{\lambda \in \Lambda} M_\lambda$ is also a flat A-module.

Proof. Recall that M being flat is equivalent to $\operatorname{Tor}_1^A(R/I,M) = 0$ for every finitely generated ideal I of A. This is equivalent to showing that the natural "multiplication" map $I \otimes_A M \to IM$ is injective for every finitely generated ideal I of A.

Let $I = (a_1,\ldots,a_n)$, and let $f : A^n \to A$ be the map given by

$$f(x_1,\ldots,x_n) = a_1x_1 + \cdots + a_nx_n,$$

and set $K = \ker f \subseteq A^n$. Since each M_λ is flat, tensoring gives us an exact sequence

$$0 \to K \otimes_A M_\lambda \to M_\lambda^n \to M_\lambda.$$

Consider an element in $\ker(I \otimes_A M \to IM)$, which can be written as

$$\sum_{i=1}^{n} a_i \otimes \xi_i$$

for some $\xi_i \in M$ for $1 \leqslant i \leqslant n$. That is,

$$\sum_{i=1}^{n} a_i \xi_i = 0 \in IM.$$

We can further write $\xi_i = (\xi_i^\lambda)_{\lambda \in \Lambda}$. Hence, for each $\lambda \in \Lambda$,

$$\sum_{i=1}^{n} a_i \xi_i^\lambda = 0 \quad \text{in } M_\lambda.$$

Hence,

$$\left(\xi_1^\lambda, \dots, \xi_n^\lambda\right) \in \ker\left(M_\lambda^n \to M_\lambda\right) = \operatorname{im}\left(K \otimes_A M_\lambda \to M_\lambda^n\right).$$

Since A is Noetherian, K is a finite A-module generated by some $\beta_1, \dots, \beta_r \in K$ and write

$$\beta_i = \left(b_1^i, \dots, b_n^i\right) \in K \subseteq A^n$$

for $1 \leqslant i \leqslant r$. Now, $\left(\xi_1^\lambda, \dots, \xi_n^\lambda\right)$ is the image of some

$$\sum_{i=1}^{r} \beta_i \otimes \eta_i^\lambda \in K \otimes_A M_\lambda$$

for some $\eta_i^\lambda \in M_\lambda$ for $1 \leqslant i \leqslant r$ and $\lambda \in \Lambda$. Therefore,

$$\sum_{i=1}^{r} \left(b_1^i, \dots, b_n^i\right) \otimes \eta^\lambda \longmapsto \left(\sum_{i=1}^{r} b_1^i \eta_i^\lambda, \dots, \sum_{i=1}^{r} b_n^i \eta_i^\lambda\right) = \left(\xi_1^\lambda, \dots, \xi_n^\lambda\right),$$

so that

$$\xi_i^\lambda = \sum_{j=1}^{r} b_i^j \eta_j^\lambda$$

for $1 \leqslant i \leqslant n$ and $\lambda \in \Lambda$. Further, since $\beta_j \in K$, we have

$$\sum_{i=1}^{n} a_i b_i^j = 0 \quad \text{for } 1 \leqslant j \leqslant r.$$

Setting $\eta_i = (\eta_i^\lambda)_{\lambda \in \Lambda} \in M$ for $1 \leqslant i \leqslant r$, we have

$$\begin{aligned}
\sum_{i=1}^{n} a_i \otimes \xi_i &= \sum_{i=1}^{n} a_i \otimes \left(\sum_{j=1}^{r} b_i^j \eta_j\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{r} a_i \otimes b_i^j \eta_j \\
&= \sum_{j=1}^{r} \left(\sum_{i=1}^{n} a_i \otimes b_i^j\right) \otimes \eta_j \\
&= 0,
\end{aligned}$$

thereby completing the proof. ■

Remark 2.10. A ring is said to be *coherent* if every finitely generated ideal is finitely presented. We note that Theorem 2.9 holds even for coherent rings with the same proof, since the Noetherian-ness of A was used only to conclude the finiteness of K, which also follows from the fact that the kernel of a surjective homomorphism from a finitely generated module to a finitely presented module is again finitely generated.

§3 Injective Modules

Definition 3.1. An A-module M is said to be *injective* if the (contravariant) functor $\operatorname{Hom}_A(-,M): \mathfrak{Mod}_A^{op} \to \mathfrak{Mod}_A$ is exact.

Theorem 3.2 (Baer's Criterion). An A-module E is injective if and only if for every ideal $\mathfrak{a} \trianglelefteq A$, every A-linear map $\mathfrak{a} \to E$ can be extended to an A-linear map $A \to E$.

Proof. The forward direction is tautological. We prove the converse. Suppose $N \leqslant M$ are A-modules and $\alpha : N \to E$ is an A-linear map. We shall extend α to a map $M \to E$.

Let Σ be the collection of all pairs (N', α') where $N \leqslant N' \leqslant M$ and $\alpha' : N' \to E$ is A-linear such that $\alpha'|_N = \alpha$. Using a standard Zorn argument, Σ admits a maximal element $\alpha' : N' \to E$ extending α. We contend that $N' = M$.

Suppose not. Then choose some $x \in M \setminus N'$ and let $\mathfrak{a} = (N' :_A x) \trianglelefteq A$. Consider the composite map $\mathfrak{a} \xrightarrow{x} N' \xrightarrow{\alpha'} E$, which extends to a map $f : A \to E$ and set $N'' = N' + Ax \leqslant M$. Define $\alpha'' : N'' \to E$ by

$$\alpha''(n' + ax) = \alpha'(n') + f(a).$$

This is well defined, for if $n_1' + a_1 x = n_2' + a_2 x$, then $(a_1 - a_2)x = n_2' - n_1'$, i.e. $(a_1 - a_2) \in \mathfrak{a}$ and hence,

$$f(a_1 - a_2) = \alpha'((a_1 - a_2)x) = \alpha'(n_2' - n_1').$$

But note that $(N', \alpha') < (N'', \alpha'')$ in Σ, a contradiction. Thus $N' = M$ and we are done. ∎

Corollary. Let A be a noetherian ring. If $\{E_i : i \in I\}$ is a collection of injective A-modules, then $E = \bigoplus_{i \in I} E_i$ is an injective A-module.

Proof. Let $\mathfrak{a} \trianglelefteq A$ and $f : \mathfrak{a} \to E$ be A-linear. Note that $\mathfrak{a} = (a_1, \ldots, a_n)$ is finitely generated, and each $f(a_i)$ has support contained in a finite subset of I. Thus, $f(\mathfrak{a})$ is contained in a direct sum of a finite subset of $\{E_i : i \in I\}$. But note that a finite direct sum of injectives in injective over any ring, and hence, f can be extended to all of A, thereby completing the proof. ∎

Corollary. Let A be a PID. An A-module E is injective if and only if it is divisible.

Proof. Immediate from Theorem 3.2. ∎

§§ Injective Hulls

Definition 3.3. Let $M \leqslant E$ be A-modules. Then E is said to be an *essential extension* of M if every non-zero submodule of E intersects M non-trivially. We denote this by $M \leqslant_e E$.

Remark 3.4. The above is equivalent to requiring that for every $x \in E \setminus \{0\}$, there is an $a \in A \setminus \{0\}$ such that $ax \in M \setminus \{0\}$.

We note some trivial properties of essential extensions before proceeding.

Proposition 3.5. Let $L \leqslant M \leqslant N$ be A-modules. Then

$$L \leqslant_e M \text{ and } M \leqslant_e N \iff L \leqslant_e N.$$

Proof. Straightforward. ∎

PROPOSITION 3.6. Let $M \leqslant E$ be A-modules. Consider the set

$$\mathscr{E} = \{N \leqslant E : M \leqslant_e N\}.$$

Then $\mathscr{E}$ has a maximal element.

Proof. Standard application of Zorn's lemma. ■

PROPOSITION 3.7. If $N_1 \leqslant_e M_1$ and $N_2 \leqslant_e M_2$, then $N_1 \oplus N_2 \leqslant_e M_1 \oplus M_2$.

Proof. Trivial. ■

REMARK 3.8. Before we proceed, we make an important observation. Suppose $M \leqslant_e N$ and suppose there is a commutative diagram:

$$\begin{array}{ccc} N & & \\ \uparrow & \overset{f}{\searrow} & \\ M & \hookrightarrow & E. \end{array}$$

We claim that f is injective. Indeed, due to the commutativity of the diagram, $\ker f \cap M = 0$, but since $M \leqslant_e N$, we have that $\ker f = 0$.

DEFINITION 3.9. Let $M \leqslant E$ be A-modules. Then E is said to be an *injective hull* of M if E is an injective A-module and $M \leqslant_e E$. It is customary to denote E by $E_A(M)$.

PROPOSITION 3.10. Suppose $M \leqslant E$ and $N \leqslant F$ are A-modules such that E and F are injective hulls of M and N respectively. Then $E \oplus F$ is an injective hullof $M \oplus N$.

Proof. Obviously $E \oplus F$ is injective and due to the preceding result, an essential extension of $M \oplus N$. The conclusion follows. ■

PROPOSITION 3.11. An A-module E is injective if and only if E has no proper essential extensions.

Proof. Suppose E were injective and $E \leqslant_e M$. Then, there is a submodule N of M such that $M = E \oplus N$. If N were non-trivial, then it would intersect E trivially, thus N must be trivial and $E = M$.

Conversely, suppose E has no proper essential extensions. There is an injective module I such that $E \hookrightarrow I$. We shall show that E is a direct summand of I. Indeed, consider the collection

$$\Sigma = \{N \leqslant I : E \cap N = 0\}.$$

A standard application of Zorn's lemma furnishes a maximal element N of Σ. Note that if M is a submodule of I properly containing N, then $E \cap M \neq 0$. The canonical projection $I \twoheadrightarrow I/N$ restricts to an injective map on E and any submodule of I/N is of the form M/N for some M containing N. Thus, it follows that $E \hookrightarrow I/N$ is an essential extension. But since E does not admit any proper essential extensions, we must have that the aforementioned map is surjective, that is, $E + N = I$, whence $E \oplus N = I$ and hence, E is injective. ■

THEOREM 3.12. Let $M \leqslant E$ be A-modules. The following are equivalent:

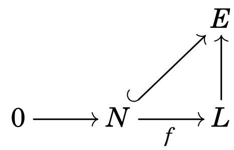
(a) E is an injective hull of M.

(b) E is a minimal injective A-module containing M.

(c) E is a maximal essential extension of M.

Proof. $(a) \implies (b)$ Suppose I is an injective module such that $M \leqslant I \leqslant E$. Since $M \leqslant_e E$, we have that $I \leqslant_e E$. But due to Proposition 3.11, we see that $I = E$.

$(b) \implies (c)$ Let $N \leqslant E$ be a maximal element of $\{N \leqslant E : M \leqslant_e N\}$. We contend that N has no proper essential extensions. Suppose $f : N \hookrightarrow L$ is an essential extension. Then, there is a map $L \to E$ making



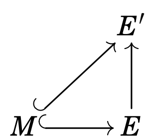
commute. We claim that the map $L \to E$ is injective. Indeed, if $0 \neq x \in L$ maps to 0, then there is an $0 \neq a \in A$ such that $0 \neq ax \in f(N)$. But since $N \hookrightarrow E$, we have that $ax = 0$, a contradiction. Thus, in E, $L = N$, since N has no proper essential extensions in E. Consequently, N has no proper essential extensions, that is, N is injective, whence $N = E$.

$(c) \implies (a)$ Injectivity follows from the fact that E has no proper essential extensions due to maximality. ■

THEOREM 3.13. Let M be an A-module. Then there exists an injective hull $M \hookrightarrow E$, which is unique up to isomorphism.

Proof. Let I be an injective module such that $M \hookrightarrow I$. Using $(b) \implies (c)$ of the proof of Theorem 3.12, we see that a maximal essential extension E of M contained in I is an injective hull.

It remains to establish uniqueness. Suppose $M \hookrightarrow E'$ is another injective hull. Then, there is a commutative diagram



with the induced map $E \to E'$ injective as argued in the preceding proof. The maximality of essentialness and transitivity of essentialness both imply that $E \to E'$ must be an isomorphism. ■

THEOREM 3.14 (CANTOR-SCHRÖDER-BERNSTEIN). If M and N are injective A-modules with injective A-linear maps $M \hookrightarrow N$ and $N \hookrightarrow M$, then $M \cong N$.

Proof. We may suppose that $N \leqslant M$, whence there is a submodule P of M such that $M = N \oplus P$ where P is injective too. Let $f : M \to N$ be an injective A-linear map.

Note first that if $x_0 + f(x_1) + \cdots + f^{(n)}(x_n) = 0$ where $x_i \in P$, then all $x_i = 0$. Indeed, $f(x_1) + \cdots + f^{(n)}(x_n) \in \operatorname{im}(f) \subseteq N$ and $x_0 \in P$, whence $x_0 = 0$. Since f is injective, we have $x_1 + \cdots + f^{(n-1)}(x_n) = 0$. Working downwards, we have our conclusion.

Now, set $X = P \oplus f(P) \oplus f^{(2)}(P) \oplus \cdots \subseteq M$ and let $E = E_A(f(X)) \subseteq N$ an injective hull. Write $N = E \oplus Q$. Since $X = P \oplus f(X)$, we have

$$E(X) \cong E(P \oplus f(X)) \cong E(P) \oplus E(f(X)) \cong P \oplus E.$$

On the other hand, since f is injective,

$$E(X) \cong E(f(X)) = E \implies P \oplus E \cong E.$$

Consequently,

$$M = N \oplus P = Q \oplus E \oplus P \cong Q \oplus E \cong N,$$

thereby completing the proof. ■

PROPOSITION 3.15. Let A be a noetherian ring and M an A-module. Then $\mathrm{Ass}_A(E(M)) = \mathrm{Ass}_A(M)$. In particular, $E(A/\mathfrak{p}) = \{\mathfrak{p}\}$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$.

Proof. Since $M \hookrightarrow E(M)$, we have that $\mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(E(M))$. Conversely, suppose $\mathfrak{p} \in \mathrm{Ass}_A(E(M))$, that is, $R/\mathfrak{p} \hookrightarrow E(M)$ and identify $R/\mathfrak{p}$ with a submodule of $E(M)$. Since $M \leqslant_e E(M)$, $(R/\mathfrak{p}) \cap M \neq 0$. Choosing a non-zero x in the intersection, we have that $\mathrm{Ann}_A(x) = \mathfrak{p}$, that is, $\mathfrak{p} \in \mathrm{Ass}_A(M)$. This completes the proof. ■

DEFINITION 3.16. A nonzero A-module M is said to be *decomposable* if there are nonzero submodules $N_1, N_2 \leqslant M$ such that $M = N_1 \oplus N_2$. An A-module that is not decomposable is said to be *indecomposable*.

THEOREM 3.17 (MATLIS). Let A be a noetherian ring and M an A-module. Then,

(a) E is an indecomposable injective A-module if and only if $E \cong E(A/\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Spec}(A)$.

(b) $E_A(A/\mathfrak{p}) \ncong E(A/\mathfrak{q})$ if $\mathfrak{p} \neq \mathfrak{q} \in \mathrm{Spec}(A)$.

(c) every injective A-module can be written as a direct sum of indecomposable A-modules.

Proof. (a) Suppose E is an indecomposable injective A-module and choose some $\mathfrak{p} \in \mathrm{Ass}_A(E)$. There is an injection $A/\mathfrak{p} \hookrightarrow E$, which extends to an injection (due to Remark 3.8) $E(A/\mathfrak{p}) \hookrightarrow E$. Since E is indecomposable, $E \cong E(A/\mathfrak{p})$.

Conversely, we must show that $E = E(A/\mathfrak{p})$ is indecomposable. Suppose $E = E_1 \oplus E_2$. The map $A/\mathfrak{p} \hookrightarrow E_1 \oplus E_2$ sends $\overline{1} \in A/\mathfrak{p}$ to some $(x_1, x_2) \in E_1 \oplus E_2$. Then,

$$\mathfrak{p} = \mathrm{Ann}_A((x_1, x_2)) = \mathrm{Ann}_A(x_1) \cap \mathrm{Ann}_A(x_2),$$

whence, we may suppose without loss of generality that $\mathfrak{p} = \mathrm{Ann}_A(x_1)$. Consequently, the composition $A/\mathfrak{p} \hookrightarrow E \twoheadrightarrow E_1$ is injective. This means that $E \twoheadrightarrow E_1$ is a lift of an injection $A/\mathfrak{p} \hookrightarrow E_1$, whence $E \twoheadrightarrow E_1$ must be injective (due to Remark 3.8), that means $E_2 = 0$, as desired.

(b) Follows from the fact that $\mathrm{Ass}_A(E(A/\mathfrak{p})) = \{\mathfrak{p}\}$.

(c) This is another standard Zorn argument. Begin with the collection

$$\Sigma = \{\{E_i\}_{i \in I} : \text{each } E_i \text{ is indecomposable injective, and their sum is direct}\}.$$

Choose a maximal element $\{E_i\}_{i \in J}$ in Σ and let $I = \bigoplus_{i \in J} E_i$. Suppose $I \neq E$. Since I is injective (owing to A being noetherian), we can write $E = I \oplus E'$. Since $E' \neq 0$, it has an associated prime, $\mathfrak{p}$. We can then write $E' = E(A/\mathfrak{p}) \oplus E''$, contradicting the maximality of $\{E_i\}_{i \in J}$. This completes the proof. ■

§4 UNCATEGORIZED

§§ Eakin-Nagata Theorem

THEOREM 4.1 (FORMANEK). Let A be a ring, and B a finitely generated faithful A-module. Suppose the set of A-submodules $\Sigma = \{\mathfrak{a}B : \mathfrak{a} \trianglelefteq A\}$ has the ascending chain condition, then A is noetherian.

Proof. It suffices to show that B is a noetherian A-module since it is finitely generated and faithful. Suppose not. Then consider the collection

$$\Gamma = \{\mathfrak{a}B : \mathfrak{a} \trianglelefteq A,\ B/\mathfrak{a}B \text{ is a non-noetherian } A\text{-module}\},$$

which contains (0) and hence is non-empty. Since Σ has the ascending chain condition, so does Γ, whence, it contains a maximal element $\mathfrak{a}B$.

Replacing B by $B/\mathfrak{a}B$, we see that B is a non-noetherian A-module. This may not be faithful and hence, replace A by $A/\mathrm{Ann}_A(B)$. Then, B is a finite, non-noetherian, faithful A-module such that for every ideal $0 \neq \mathfrak{a} \triangleleft A$, $B/\mathfrak{a}B$ is a noetherian A-module.

Next, set

$$\mathfrak{M} = \{N \leqslant B : B/N \text{ is a faithful } A\text{-module}\},$$

which is non-empty, since $\{0\} \in \mathfrak{M}$. Suppose B is generated as an A-module by $b_1, \ldots, b_n$. It is not hard to argue that

$$N \in \mathfrak{M} \iff \forall\ a \in A \setminus \{0\},\ \{ab_1, \ldots, ab_n\} \not\subseteq N.$$

It follows that every chain in $\mathfrak{M}$ has a maximal element and hence Zorn's Lemma applies to furnish a maximal element $N_0 \in \Gamma$.

If B/N_0 is a noetherian A-module, then A is noetherian since B/N_0 is faithful and finite. If not, replace B with B/N_0, which is still a finite faithful A-module and satisfies:

(1) B is a non-noetherian A-module.

(2) for any ideal $0 \neq \mathfrak{a} \trianglelefteq A$, $B/\mathfrak{a}B$ is a noetherian A-module.

(3) for any submodule $0 \neq N$ of B, B/N is not faithful as an A-module.

Now, let N be a non-zero submodule of B. Due to (3), there is a $0 \neq a \in A$ such that $aB \subseteq N$. Due to (2), B/aB is a noetherian A-module with N/aB as a submodule. Thus, N/aB is a noetherian, in particular, a finite A-module. Since aB is also finite as an A-module, we have that N is a finite A-module. Hence, B is a noetherian A-module, which is absurd. This completes the proof. ■

THEOREM 4.2 (EAKIN-NAGATA). Let $A \subseteq B$ be an extension of rings such that B is a finite A-module. If B is a noetherian ring, then so is A.

Proof. Note that B is a finite, faithful A-module, since $1 \in B$. The conclusion follows from Theorem 4.1. ■