Analytic Number Theory

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§1 Some Background on Sequences and Series

Theorem 1.1 (Summation By Parts). Let (a_n) and (b_n) be two sequences. Put

$$A_{m,n} = \sum_{k=m}^{n} a_k \quad \text{and} \quad S_{m,n} = \sum_{k=m}^{n} a_k b_k.$$

Then, for $m < n$,

$$S_{m,n} = \sum_{k=m}^{n-1} A_{m,k}(b_k - b_{k+1}) + A_{m,n} b_n.$$

Theorem 1.2 (Partial Summation Formula). Let $(a_n)_{n=1}^{\infty}$ be a sequence of complex numbers and $f : [1, x] \to \mathbb{C}$ a continuously differentiable function. Set

$$A(t) = \sum_{1 \leqslant n \leqslant t} a_n.$$

Then,

$$\sum_{1 \leqslant n \leqslant x} a_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t)\, dt.$$

Proof. Suppose x is a natural number.

$$\begin{aligned}
\sum_{1 \leqslant n \leqslant x} a_n f(n) &= \sum_{1 \leqslant n \leqslant x} \left(A(n) - A(n-1)\right) f(n) \\
&= \sum_{1 \leqslant n \leqslant x} A(n) f(n) - \sum_{0 \leqslant n \leqslant x-1} A(n) f(n+1) \\
&= A(x) f(x) - \sum_{0 \leqslant n \leqslant x-1} A(n) \int_n^{n+1} f'(t)\, dt \\
&= A(x) f(x) - \sum_{0 \leqslant n \leqslant x-1} \int_n^{n+1} A(t) f'(t)\, dt \\
&= A(x) f(x) - \int_0^x A(t) f'(t) \\
&= A(x) f(x) - \int_1^x A(t) f'(t)\, dt.
\end{aligned}$$

If x is not a natural number, note the equality

$$A(x)\left(f(x)-f(\lfloor x\rfloor)\right)=\int_{\lfloor x\rfloor}^{x} A(t)f'(t)\,dt. \qquad \blacksquare$$

COROLLARY (PARTIAL SUMS OF DIRICHLET SERIES). Take $f(t)=1/t^s$ to obtain (for $x\geqslant 1$)

$$\sum_{1\leqslant n\leqslant x}\frac{a_n}{n^s}=\frac{A(x)}{x^s}+s\int_1^x \frac{A(t)}{t^{s+1}}\,ds.$$

This is often called *Abel's Summation Formula*.

EXAMPLE 1.3. In Abel's formula, set $a_n=1$ for all n and $s=1$. Then,

$$\sum_{1\leqslant n\leqslant x}=\frac{\lfloor x\rfloor}{x}+\int_1^x\frac{\lfloor t\rfloor}{t^2}\,dt.$$

The integral is bounded by

$$\int_1^x\frac{1}{t}\,dt=\log x.$$

It follows that

$$\sum_{1\leqslant n\leqslant x}\frac{1}{n}=\log x+O(1).$$

EXAMPLE 1.4. As a consequence of the above example,

$$\sum_{\leqslant n\leqslant x}d(n)=\sum_{1\leqslant n\leqslant x}\left\lfloor\frac{x}{n}\right\rfloor=x\sum_{1\leqslant n\leqslant x}\frac{1}{n}+O(x)=x\log x+O(x).$$

Next, we elucidate *Dirichlet's Hyperbola Method* using a theorem due to Dirichlet.

THEOREM 1.5 (DIRICHLET).

$$\sum_{1\leqslant n\leqslant x}d(n)=x\log x+(2\gamma-1)x+O(\sqrt{x}).$$

Proof. ■ Add in

§2 ELEMENTARY RESULTS ON PRIME NUMBERS

DEFINITION 2.1. The two *Chebyshev functions* are defined as

$$\psi(x)=\sum_{p\leqslant x}\Lambda(x)\quad\text{and}\quad\vartheta(x)=\sum_{p\leqslant x}\log p,$$

for $x>0$.

PROPOSITION 2.2.
$$\Lambda(x) = \sum_{m=1}^{\infty} \vartheta(x^{1/m}) = \sum_{m\leqslant \log_2 x} \vartheta(x^{1/m}).$$

Proof. We have
$$\psi(x) = \sum_{n\leqslant x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p^m\leqslant x} \log p = \sum_{m=1}^{\infty} \sum_{p\leqslant x^{1/m}} \log p = \sum_{m=1}^{\infty} \vartheta(x^{1/m}). \qquad \blacksquare$$

PROPOSITION 2.3.
$$0 \leqslant \frac{\psi(x)-\vartheta(x)}{x} \leqslant \frac{(\log x)^2}{2\sqrt{x}\log 2}.$$

Proof. We have
$$\frac{\psi(x)-\vartheta(x)}{x} \leqslant \frac{1}{x}\sum_{2\leqslant m\leqslant \log_2 x} \vartheta(x^{1/m}) \leqslant \frac{1}{x}\sum_{2\leqslant m\leqslant \log_2 x} x^{1/m}\log x^{1/m} \leqslant \frac{(\log x)^2}{2\sqrt{x}\log 2}. \qquad \blacksquare$$

LEMMA 2.4. For $x \geqslant 2$, we have
$$\vartheta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,dt,$$
and
$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t\log^2 t}\,dt.$$

Proof. Both follow from Theorem 1.2. ■

THEOREM 2.5. The following are equivalent:

(a) $\displaystyle\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1,$

(b) $\displaystyle\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1,$

(c) $\displaystyle\lim_{x\to\infty} \frac{\psi(x)}{x} = 1.$

Proof. Suppose (a) holds. Using the preceding lemma, we have
$$\frac{\vartheta(x)}{x} = \frac{\pi(x)\log x}{x} - \frac{1}{x}\int_2^x \frac{\pi(t)}{t}\,dt.$$

But (a) implies $\pi(x) = O\left(\frac{x}{\log x}\right)$, i.e. there is an $M > 0$ such that $\pi(x) \leqslant \frac{Mx}{\log x}$. Hence,
$$\frac{1}{x}\int_2^x \frac{\pi(t)}{t}\,dt = M\frac{1}{x}\int_2^x \frac{dt}{\log t} = \frac{M}{x}\left(\int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_{\sqrt{x}}^x \frac{dt}{\log t}\right) \leqslant \frac{M}{x}\left(\frac{\sqrt{x}-2}{\log\sqrt{x}} + \frac{x-\sqrt{x}}{\log x}\right) \to 0$$

as $x \to \infty$.

Conversely, suppose (b) holds. Using the preceding lemma, we have

$$\frac{\pi(x)\log x}{x} = \frac{\vartheta(x)}{x} - \frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t\log^2 t}\,dt.$$

But (a) implies the existence of a constant $M > 0$ such that $\vartheta(x) \leqslant Mx$. Hence,

$$\frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{\log^2 t}\,dt \leqslant \frac{M\log x}{x}\int_2^x \frac{dt}{\log^2 t} = \frac{M\log x}{x}\left(\int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t}\right) \leqslant \frac{M\log x}{x}\left(\frac{\sqrt{x}-2}{\log^2\sqrt{x}} + \cdots\right.$$

and the conclusion follows.

Finally, the equivalence of (b) and (c) follows from Proposition 2.3. ■

§3 Dirichlet Characters and Gauss Sums

Definition 3.1. A *Dirichlet character modulo n* is a group homomorphism $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ which is extended by 0 to $\mathbb{Z}/n\mathbb{Z}$ and extended periodically to all of $\mathbb{Z}$.

Definition 3.2. Let χ be a Dirichlet character modulo n. Define its *Gauss sums* as

$$G(m,\chi) = \sum_{r \bmod n} \chi(r)\exp\left(\frac{2\pi i m}{n} r\right).$$

Lemma 3.3. If χ is any Dirichlet character modulo n, then

$$G(m,\chi) = \overline{\chi}(m)G(1,\chi),$$

whenever $(m,n) = 1$.

Proof. We have

$$\begin{aligned} G(m,\chi) &= \sum_{r \bmod n} \overline{\chi}(m)\chi(m)\chi(r)\exp\left(\frac{2\pi i m}{n} r\right) \\ &= \overline{\chi}(m)\sum_{r \bmod n} \chi(mr)\exp\left(\frac{2\pi i m r}{n}\right) \\ &= \overline{\chi}(m)G(1,\chi), \end{aligned}$$

where the last equality follows from the fact that $(m,n) = 1$. ■

Definition 3.4. The Gauss sum $G(m,\chi)$ is said to be *separable* if

$$G(m,\chi) = \overline{\chi}(m)G(1,\chi).$$

We have seen that $G(m,\chi)$ is separable when $(m,n) = 1$.

PROPOSITION 3.5. Let χ be a Dirichlet character modulo n. Then, the Gauss sum $G(m,\chi)$ is separable for every m if and only if $G(m,\chi) = 0$ whenever $(m,n) > 1$.

Proof. Immediate from the definition. ■

THEOREM 3.6. Let χ be a Dirichlet character modulo n. If $G(m,\chi)$ is separable for every m, then

$$|G(1,\chi)|^2 = n,$$

Proof. We have

$$\begin{aligned}
|G(1,\chi)|^2 = G(1,\chi)\overline{G(1,\chi)} &= \sum_{m=1}^{n} G(1,\chi)\overline{\chi}(m) \exp\left(-\frac{2\pi i}{n}m\right) \\
&= \sum_{m=1}^{n} G(m,\chi) \exp\left(-\frac{2\pi i m}{n}\right) \\
&= \sum_{m=1}^{n}\sum_{k=1}^{n} \chi(k) \exp\left(\frac{2\pi i m}{n}k\right) \exp\left(-\frac{2\pi i m}{n}\right) \\
&= \sum_{k=1}^{n} \chi(k) \sum_{m=1}^{n} \exp\left(\frac{2\pi i(k-1)}{n}m\right) \\
&= n\chi(1) = n.
\end{aligned}$$

■

LEMMA 3.7. Let χ be a Dirichlet character modulo n and suppose $G(m,\chi) \neq 0$ for some m with $(m,n) > 1$. Then, χ is not primitive.

Proof. Let $q = (m,n)$ and set $d = n/q$. Choose any a satisfying $(a,n) = 1$ and $a \equiv 1 \bmod d$. We have

$$G(m,\chi) = \sum_{r \bmod n} \chi(r)e_n(mr) = \sum_{r \bmod n} \chi(ar)e_n(amr) = \chi(a) \sum_{r \bmod n} \chi(r)e_n(amr)$$

Note that $a = 1 + bd$ for some integer b. Hence,

$$\frac{amr}{n} = \frac{mr + mrbd}{n} = \frac{mr}{n} \bmod 1.$$

Consequently,

$$G(m,\chi) = \chi(a)G(m,\chi).$$

This shows that $\chi(a) = 1$. We have shown that for any a satisfying $a \equiv 1 \bmod d$ and $(a,n) = 1$, $\chi(a) = 1$ and since $d < n$, χ cannot be primitive. ■

THEOREM 3.8. Let χ be a primitive Dirichlet character modulo n. Then, we have

(a) $G(m,\chi) = 0$ whenever $(m,n) > 1$.

(b) $G(m,\chi)$ is separable for every m.

(c) $|G(m,\chi)|^2 = n$.

§§ Quadratic Reciprocity using Gauss Sums

If p is a prime, there is a unique non-principal quadratic character modulo p, which is given by

$$\chi(r) = \left(\frac{r}{p}\right).$$

THEOREM 3.9. If p is an odd prime and χ is the unique non-principal quadratic character modulo p, then

$$G(1,\chi)^2 = \left(\frac{-1}{p}\right) p.$$

Proof. We have

$$G(1,\chi)^2 = \sum_{r=1}^{p-1}\sum_{s=1}^{p-1} \chi(r)\chi(s)e_p(r+s).$$

For each pair (r,s), there is a unique t modulo p satisfying $tr \equiv s \bmod p$. Therefore, we can write the sum as

$$\sum_{t=1}^{p-1}\sum_{r=1}^{p-1} \chi(t)e_p(r(1+t)) = \sum_{t=1}^{p-1}\chi(t)\sum_{r=1}^{p-1} e_p(r(1+t)) = -\sum_{t=1}^{p-2}\chi(t) + (p-1)\chi(p-1).$$

Since

$$\sum_{t=1}^{p-1}\chi(t) = 0,$$

the proof is complete. ∎

Let p and q be distinct odd primes. From the above theorem, we have

$$G(1,\chi)^{q-1} \equiv \left(\frac{-1}{p}\right)^{\frac{q-1}{2}} \left(\frac{p}{q}\right) \bmod q \equiv (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right) \bmod q.$$

THEOREM 3.10. Let p and q be distinct odd primes and χ the non-principal quadratic character modulo p, then

$$G(1,\chi)^{q-1} = \left(\frac{q}{p}\right) \underset{r_1+\cdots+r_q\equiv q \bmod p}{\sum_{r_1}\cdots\sum_{r_q}} \left(\frac{r_1\cdots r_q}{p}\right).$$

Proof. The Gauss sum $G(n,\chi)$ is periodic with period p and hence, has a finite Fourier expansion,

$$G(n,\chi)^q = \sum_{m=1}^{p} a_q(m)e_p(mn),$$

where the coefficients can be recovered as

$$a_q(m) = \frac{1}{p}\sum_{n=1}^{p} G(n,\chi)^q e_p(-mn).$$

From the definition, we have

$$G(n,\chi)^q = \left(\sum_{r \bmod p} \chi(r)e_p(nr)\right)^q = \sum_{r_1 \bmod p} \cdots \sum_{r_q \bmod p} \chi(r_1 \cdots r_q)e_p(n(r_1+\cdots+r_q)).$$

Hence,

$$a_q(m) = \frac{1}{p} \sum_{r_1 \bmod p} \cdots \sum_{r_q \bmod p} \chi(r_1 \dots r_q) \sum_{n=1}^{p} e_p(n(r_1+\cdots+r_q-m)).$$

The innermost sum takes a non-zero value if and only if $r_1+\cdots+r_q \equiv m \bmod p$. As a result, we have

$$a_q(m) = \underset{r_1+\cdots+r_q \equiv m \bmod p}{\sum_{r_1} \cdots \sum_{r_q}} \chi(r_1 \dots r_q).$$

On the other hand, $G(n,\chi)$ is separable and hence, we have

$$\begin{aligned}
a_q(M) &= \frac{1}{p}G(1,\chi)^q \sum_{n=1}^{p} \chi(n)^q e_p(-mn) = \frac{1}{p}G(1,\chi)^q \sum_{n=1}^{p} \chi(n)e_p(-mn) \\
&= \frac{1}{p}G(1,\chi)^q G(-m,\chi) = \frac{1}{p}G(1,\chi)^q \chi(m)G(-1,\chi) \\
&= \frac{1}{p}G(1,\chi)^q \chi(m)\overline{G(1,\chi)} = \chi(m)G(1,\chi)^{q-1}.
\end{aligned}$$

Therefore,

$$G(1,\chi)^{q-1} = \chi(m) \underset{r_1+\cdots+r_q \equiv m \bmod p}{\sum_{r_1} \cdots \sum_{r_q}} \chi(r_1 \dots r_q).$$

Taking $m = q$, we have the desired conclusion. ■

PROOF OF QUADRATIC RECIPROCITY. Putting together the last two theorems,

$$(-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right) \equiv \left(\frac{q}{p}\right) \underset{r_1+\cdots+r_q \equiv q \bmod p}{\sum_{r_1} \cdots \sum_{r_q}} \left(\frac{r_1 \dots r_q}{p}\right) (\bmod q)$$

We can break the sum on the right into equivalence classes corresponding to multisets $(r_1, \dots, r_q)$. If all the r_i's are not equal, then the number of distinct permutations of this multiset is divisible by q and hence, the only term that survives on the right is when all the r_i's are equal to 1. This gives the desired conclusion. ■

§4 Dirichlet Series

A *Dirichlet series* is a "formal sum" of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

where $s \in \mathbb{C}$ and $f : \mathbb{N} \to \mathbb{C}$ is an arithmetic function. The first thing to study is its convergence. As is customary, we shall write $s = \sigma + it$.

Theorem 4.1. Suppose the series $\sum |f(n)n^{-s}|$ does not converge for all s or diverge for all s. Then there is a real number σ_a called the *abscissa of absolute convergence* such that the series $\sum f(n)n^{-s}$ converges absolutely if $\sigma > \sigma_a$ but does not converge absolutely if $\sigma < \sigma_a$.

Proof. Omitted on account of its obviousness. ■

Remark 4.2. If the Dirichlet series converges absolutely everywhere, we set $\sigma_a = -\infty$ and if it converges absolutely nowhere, we set $\sigma_a = \infty$.

We set

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

which is a well defined function on the half plane $\sigma > \sigma_a$.

Lemma 4.3. If $N \geqslant 1$ and $\sigma \geqslant c > \sigma_a$,

$$\left| \sum_{n=N}^{\infty} f(n)n^{-s} \right| \leqslant N^{-(\sigma-c)} \sum_{n=N}^{\infty} |f(n)|n^{-c}.$$

Proof. Indeed,

$$\begin{aligned} \left| \sum_{n=N}^{\infty} f(n)n^{-s} \right| &\leqslant \sum_{n=N}^{\infty} |f(n)|n^{-\sigma} \\ &\leqslant \sum_{n=N}^{\infty} |f(n)|n^{-c}N^{-(\sigma-c)}. \end{aligned}$$

■

Proposition 4.4.

$$\lim_{\sigma\to\infty} F(\sigma + it) = f(1)$$

uniformly for $t \in \mathbb{R}$.

Proof. Immediate from the above lemma. ■

Theorem 4.5 (Uniqueness Theorem for Dirichlet Series). Given two Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

both absolutely convergent for $\sigma > \sigma_a$. If $F(s) = G(s)$ for an infinite sequence $\{s_k\}$ with $\sigma_k \to \infty$. Then, $f(n) = g(n)$ for every n.

Proof. Set $h(n) = f(n) - g(n)$ and $H(s) = F(s) - G(s)$. Then, $H(s_k) = 0$ for each k and $\sigma_k \to \infty$ as $k \to \infty$. Suppose h is not identically 0 and let N be the smallest positive integer for which $h(n) \neq 0$. Then,

$$H(s) = \frac{h(N)}{N^s} + \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Thus,

$$h(N) = N^s H(s) - N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Put $s = s_k$ to obtain

$$h(N) = -N^{s_k} \sum_{n=N+1}^{\infty} \frac{h(n)}{n^{s_k}}.$$

Choose some $c > \sigma_a$. Then, for sufficiently large k, $\sigma_k > c > \sigma_a$. Then,

$$|h(N)| = N^{\sigma_k} (N+1)^{-(\sigma_k - c)} \sum_{n=N+1}^{\infty} |h(n)| n^{-c}.$$

It follows by taking $k \to \infty$ that $h(N) = 0$. ■

Corollary. Let $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ and suppose $F(s) \neq 0$ for some s with $\sigma > \sigma_a$. Then, there is a constant $c \geqslant \sigma_a$ such that $F(s)$ does not vanish for $\sigma > c$.

Proof. Converse to the previous theorem. ■

Theorem 4.6. Consider two Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

which are absolutely convergent for $\sigma > a$ and $\sigma > b$ respectively. Then, in the half-plane where both series converge absolutely,

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s},$$

and converges absolutely. Conversely, if $F(s)G(s) = \sum \alpha(n) n^{-s}$ for a sequence $\{s_k\}$ with $\sigma_k \to \infty$ as $k \to \infty$, then $\alpha = f * g$.

Proof. The first statement follows from the fact that absolutely convergent series can be rearranged. The second statement follows from the uniqueness theorem. ∎

EXAMPLE 4.7. The zeta function is the Dirichlet series corresponding to $f \equiv \mathbb{1}$. Let $G(s)$ denote the function defined by the Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

which is absolutely convergent in the right half plane $\sigma > 1$. Then,

$$\zeta(s)G(s) = \sum_{n=1}^{\infty} \frac{(\mathbb{1} * \mu)(n)}{n^s} = 1,$$

for $\sigma > 1$. This, in turn, shows that ζ does not vanish in the right half plane $\sigma > 1$.

EXAMPLE 4.8. In the spirit of the previous example, let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative arithmetic function. Then, its Dirichlet inverse is given by $f^{-1}(n) = \mu(n)f(n)$. If σ_a denotes the abscissa of absolute convergence for the Dirichlet series corresponding to f, then the Dirichlet series corresponding to f^{-1} converges absolutely in the half plane $\sigma > \sigma_a$.

Consequently, for $\sigma > \sigma_a$, we have

$$\frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)f(n)}{n^s},$$

therefore, $F(s) \neq 0$ in the right half plane $\sigma > \sigma_a$.

In particular, for a Dirichlet character χ (modulo N), we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} = \frac{1}{L(s,\chi)} \quad \text{for } \sigma > 1.$$

EXAMPLE 4.9. Taking $f \equiv \mathbb{1}$ and $g = \lambda$, Liouville's function, we get, for $\sigma > 1$,

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(n^2)^s} = \zeta(2s).$$

That is,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \quad \text{for } \sigma > 1.$$

PROPOSITION 4.10. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative arithmetic function such that the series $\sum_{n \geqslant 1} f(n)$ is absolutely convergent. Then,

$$\sum_{n=1}^{\infty} f(n) = \prod_{p \text{ prime}} \left\{1 + f(p) + f(p^2) + \cdots\right\},$$

where the product is absolutely convergent. If f is completely multiplicative,

$$\sum_{n=1}^{\infty} = \prod_{p \text{ prime}} \frac{1}{1 - f(p)}.$$

Proof. Straightforward. Note that *absolute convergence* is necessary. ■

THEOREM 4.11 (EULER PRODUCT). Suppose the Dirichlet series $\sum_{n\geqslant 1} f(n)n^{-s}$ converges absolutely for $\sigma > \sigma_a$. If f is multiplicative, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} \left\{1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right\} \quad \text{for } \sigma > \sigma_a,$$

and if f is completely multiplicative, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - f(p)p^{-s}}.$$

EXAMPLE 4.12. Let χ be a Dirichlet character (modulo N), then

$$L(s,\chi) = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p)p^{-s}}.$$

LEMMA 4.13. Let $s_0 = \sigma_0 + it_0$ and assume that the Dirichlet series $\sum_{n\geqslant 1} f(n)n^{-s_0}$ has bounded partial sums, say

$$\left|\sum_{n\leqslant x} f(n)n^{-s_0}\right| \leqslant M,$$

for all $x \geqslant 1$. Then, for each s with $\sigma > \sigma_0 <$ we have

$$\left|\sum_{a<n\leqslant b} f(n)n^{-s}\right| \leqslant 2Ma^{\sigma_0-\sigma}\left(1 + \frac{|s-s_0|}{\sigma-\sigma_0}\right)$$

Proof. ■

Abel summation

COROLLARY. If the Dirichlet series $\sum_{n\geqslant 1} f(n)n^{-s}$ converges for $s_0 = \sigma_0 + it_0$, then it also converges for all s with $\sigma > \sigma_0$. If, on the other hand, it diverges for $s_0 = \sigma_0 + it_0$, then it diverges for all s with $\sigma < \sigma_0$.

Proof. The second statement follows from the first. To see the first statement, choose any s with $\sigma > \sigma_0$. The preceding lemma shows that there is a constant $C > 0$ such that

$$\left|\sum_{a<n\leqslant b} f(n)n^{-s}\right| \leqslant Ca^{\sigma_0-\sigma},$$

where C does not depend on a. Now, since $a^{\sigma_0-\sigma} \to 0$ as $a \to \infty$, the partial sums form a Cauchy sequence and we are done. ■

THEOREM 4.14. If the Dirichlet series $\sum_{n\geqslant 1} f(n)n^{-s}$ does not converge everywhere or diverge everywhere, then there exists a real number σ_c called the *abscissa of convergence*, such that the series converges for all s in the half plane $\sigma > \sigma_c$ and diverges for all s in the half plane $\sigma < \sigma_c$.

Proof. Omitted on account of its obviousness. ■

THEOREM 4.15. For any Dirichlet series with σ_c finite, we have

$$0 \leqslant \sigma_a - \sigma_c \leqslant 1.$$

Proof. Obviously, $\sigma_c \leqslant \sigma_a$. Now, if $\sigma > \sigma_c + 1$, then there is an $\varepsilon > 0$ such that $\sigma - \sigma_c > 1 + \varepsilon$. We can then write

$$\sum_{n \geqslant 1} \frac{|f(n)|}{n^\sigma} = \sum_{n \geqslant 1} \frac{|f(n)|}{n^{\sigma_c + \varepsilon}} \frac{1}{n^{\sigma - \sigma_c - \varepsilon}}.$$

Since the series

$$\sum_{n \geqslant 1} \frac{f(n)}{n^{\sigma_c + \varepsilon}}$$

converges, the individual terms are bounded in absolute value, say by $M > 0$. Then, we have

$$\sum_{n \geqslant 1} \frac{|f(n)|}{n^\sigma} \leqslant M \sum_{n \geqslant 1} \frac{1}{n^{\sigma - \sigma_c - \varepsilon}} < \infty.$$

Thus, $\sigma_c \leqslant \sigma_c \leqslant \sigma$. Since this inequality holds for all $\sigma > \sigma_c + 1$, we have the desired inequality. ■

PROPOSITION 4.16. Let $f : \mathbb{N} \to \mathbb{C}$ such that

$$\left| \sum_{n=1}^{N} f(n) \right| = O(N^{\sigma_0}),$$

then $\sigma_c \leqslant \sigma_0$.

Proof. Let $s = \sigma \in \mathbb{R}$ with $\sigma > \sigma_0$. Set

$$A_{m,n} = \sum_{k=m}^{n} f(k)$$

and

$$S_{m,n} = \sum_{k=m}^{n} \frac{f(k)}{k^s}.$$

Using Theorem 1.1,

$$S_{m,n} = \sum_{k=m}^{n-1} A_{m,k} \left(\frac{1}{k^s} - \frac{1}{(k+1)^s} \right) + A_{m,n} \frac{1}{n^s},$$

thus,

$$|S_{m,n}| \leqslant \sum_{k=m}^{n-1} |A_{m,k}| \left| \frac{1}{k^s} - \frac{1}{(k+1)^s} \right| + |A_{m,n}| \frac{1}{n^\sigma}.$$

According to our hypothesis, there is a constant $M > 0$ such that $A_{m,k} \leqslant Mk^{\sigma_0}$. Using the Mean Value Theorem,

$$\left|\frac{1}{k^\sigma} - \frac{1}{(k+1)^\sigma}\right| = \frac{|\sigma|}{(k+c)^{\sigma+1}} \leqslant \frac{|\sigma|}{k^{\sigma+1}}.$$

Substituting this back, we have

$$|S_{m,n}| \leqslant M|\sigma| \sum_{k=m}^{n-1} \frac{1}{k^{\sigma+1-\sigma_0}} + \frac{1}{n^{\sigma-\sigma_0}}.$$

It is easy to see that this sequence is Cauchy and hence, it converges. It follows that $\sigma_c \leqslant \sigma_0$. ■

§§ Analytic Properties of Dirichlet series

THEOREM 4.17. A Dirichlet series $\sum_{n\geqslant 1} f(n)n^{-s}$ converges uniformly on every compact subset lying in the interior of the right half plane $\sigma > \sigma_c$ and hence, defines a holomorphic function on the aforementioned right half plane.

Proof. It suffices to show uniform convergence on every compact rectangle of the form $[\alpha, \beta] \times [c, d]$ with $\alpha > \sigma_c$. First, choose a σ_0 with $\sigma_c < \sigma_0 < \alpha$. Then, using Lemma 4.13,

$$\left|\sum_{a<n\leqslant b} f(n)n^{-s}\right| \leqslant 2Ma^{\sigma_0-\sigma}\left(1 + \frac{|s-\sigma_0|}{\sigma-\sigma_0}\right).$$

There is a constant $C > 0$ such that $|s - \sigma_0| < C$ whenever s lies in the rectangle. Consequently,

$$\left|\sum_{a<n\leqslant b} f(n)n^{-s}\right| \leqslant 2Ma^{\sigma_0-\alpha}\left(1 + \frac{C}{\alpha-\sigma_0}\right),$$

for all $a \in \mathbb{N}$. This shows that the partial sums are uniformly Cauchy on the rectangle and hence, converge uniformly. This completes the proof. ■

COROLLARY. The function $F(s) := \sum_{n\geqslant 1} f(n)n^{-s}$ is analytic in the half plane $\sigma > \sigma_c$, and its derivative in the aforementioned half plane is given by

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s}.$$

EXAMPLE 4.18. For $\sigma > 1$, we have

$$\zeta'(s) = \sum_{n=1}^{\infty} \frac{\log n}{n^s},$$

where the sum is also absolutely convergent. On the other hand, recall that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

Consequently, for $\sigma > 1$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{(\mu * \log)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

THEOREM 4.19. Let F be a holomorphic function which is represented in the half plane $\sigma > c$ by the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

where c is finite. Further, suppose there is a positive integer n_0 such that $f(n) \geqslant 0$ for all $n \geqslant n_0$. If F is holomorphic in a neighborhood of c, then there is an $\varepsilon > 0$ such that the Dirichlet series converges in the half plane $\sigma > c - \varepsilon$, in other words, $\sigma_c \leqslant c - \varepsilon$.

Proof. Let $a = 1 + c$. Since F is analytic at a, it can be represented by an absolutely convergent power series about a,

$$F(s) = \sum_{k=0}^{\infty} \frac{F^{(k)}(a)}{k!}(s-a)^k,$$

whose radius of convergence is greater than 1 and hence, there is an $\varepsilon > 0$ such that $c - \varepsilon$ lies within the open disk of convergence of the aforementioned power series about a. But,

$$F^{(k)}(a) = (-1)^k \sum_{n=1}^{\infty} \frac{f(n) \log^k n}{n^a}.$$

Therefore,

$$F(s) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(a-s)^k}{k!} \frac{f(n) \log^k n}{n^a}.$$

In particular, this equality holds for $s = c - \varepsilon$. Hence,

$$F(c-\varepsilon) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(1+\varepsilon)^k}{k!} \frac{f(n) \log^k n}{n^a}.$$

The double series has *nonnegative* terms for $n \geqslant n_0$ and hence, we can interchange the order of summation.

$$F(c-\varepsilon) = \sum_{n=1}^{\infty} \frac{f(n)}{n^a} \sum_{k=0}^{\infty} \frac{(1+\varepsilon)^k \log^k n}{k!} = \sum_{n=1}^{\infty} \frac{f(n)}{n^a} n^{1+\varepsilon} = \sum_{n=1}^{\infty} \frac{f(n)}{n^{c-\varepsilon}}.$$

Thus, the Dirichlet series converges for $s = c - \varepsilon$. ■

THEOREM 4.20. Let the Dirichlet series $F(s) = \sum_{n \geqslant 1} f(n) n^{-s}$ be absolutely convergent for $\sigma > \sigma_a$ and assume that $f(1) \neq 0$. If $F(s) \neq 0$ for $\sigma > \sigma_0 \geqslant \sigma_a$, then for $\sigma > \sigma_0$, we have $F(s) = \exp(G(s))$ where

$$G(s) = \log f(1) + \sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{\log n} \frac{1}{n^s},$$

where f^{-1} is the Dirichlet inverse of f and $f'(n) = f(n) \log n$. Further, this Dirichlet series is absolutely convergent in the half plane $\sigma > \sigma_0$.

Proof. Since F does not vanish in the right half plane $\sigma > \sigma_0$, there is a holomorphic function G such that $F(s) = \exp(G(s))$. We have $G'(s) = F'(s)/F(s)$ for all $\sigma > \sigma_0$. But we already know

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s} \quad \text{and} \quad \frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^s},$$

for $\sigma > \sigma_0$ and the convergence is absolute there. Thus,

$$G'(s) = -\sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{n^s}.$$

Therefore,

$$G(s) = C + \sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{\log n} \frac{1}{n^s},$$

since the Dirichlet series for G converges absolutely in $\sigma > \sigma_0$ and upon differentiating, we obtain the Dirichlet series for G'. To determine the constant, use

$$f(1) = \lim_{\sigma \to \infty} F(\sigma + it) = e^C.$$

This completes the proof. ■

EXAMPLE 4.21. We have shown earlier that $\zeta(s)$ does not vanish on the half plane $\sigma > 1$. Therefore, it has a "logarithm" here, given by

$$G(s) = \sum_{n=2}^{\infty} \frac{(\mathbb{1}' * \mathbb{1}^{-1})(n)}{\log n} \frac{1}{n^s}.$$

Where $\mathbb{1}^{-1} = \mu$ and $\mathbb{1}' = \log$. Thus,

$$\log \zeta(s) = G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \frac{1}{n^s},$$

on the half plane $\sigma > 1$. Unraveling the definition of the von Mangoldt function,

$$G(s) = \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \quad \text{for } \sigma > 1.$$

EXAMPLE 4.22. Similarly, given a completely multiplicative arithmetic function $f : \mathbb{N} \to \mathbb{C}$, if $F(s) = \sum_{n \geqslant 1} f(n)n^{-s}$ denotes the Dirichlet series, that is non vanishing in $\sigma > \sigma_0 \geqslant \sigma_a$, then $F(s) = \exp(G(s))$ in $\sigma > \sigma_0$ and

$$G(s) = \sum_{n=2}^{\infty} \frac{f(n)\Lambda(n)}{\log n} \frac{1}{n^s} = \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{f(p)^m}{mp^{ms}}.$$

§§ Dirichlet's Theorem on Primes in Arithmetic Progressions

Our goal, in this subsection, is to show that whenever $(a, N) = 1$, there are infinitely many primes $p \equiv a \bmod N$. Henceforth, all Dirichlet characters will be modulo N. The principal character (modulo N) will be denoted by $\mathbb{1}$.

For each character χ, define

$$l_1(s, \chi) = \sum_{p \text{ prime}} \frac{\chi(p)}{p^s}.$$

This is a Dirichlet series, which is absolutely convergent and holomorphic in the half plane $\sigma > 1$. Also, define

$$l(s, \chi) = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{\chi(p)^n}{np^{ns}},$$

and we have seen in the previous section that $l(s, \chi)$ is absolutely convergent for $\sigma > 1$, is holomorphic there and $\exp(l(s, \chi)) = L(s, \chi)$ for $\sigma > 1$.

Proposition 4.23. Let $R(s, \chi) = l(s, \chi) - l_1(s, \chi)$. Then, R is a Dirichlet series that is absolutely convergent and holomorphic for $\sigma > 1/2$.

Proof. The difference of two Dirichlet series is a Dirichlet series. Let $\sigma > 1/2$. Then,

$$|R(s, \chi)| \leqslant \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{1}{np^{n\sigma}}$$

■

Complete this section

§5 Analytic Continuation for $\zeta(s)$ and $L(s, \chi)$

Definition 5.1. For $\sigma > 1$ and $0 < a \leqslant 1$, define the *Hurwitz Zeta Function* $\zeta(s, a)$ as

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

The sum is absolutely convergent in the half plane $\sigma > 1$ and defines a holomorphic function there.

Theorem 5.2. For $\sigma > 1$, we have the integral representation

$$\Gamma(s)\zeta(s, a) = \int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1 - e^{-x}}\, dx.$$

Proof. First, let $s > 1$ be real. Then, the Monotone Convergence Theorem gives

$$\int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1 - e^{-x}}\, dx = \sum_{n=0}^{\infty} \int_0^{\infty} x^{s-1}e^{-(n+a)x} = \sum_{n=0}^{\infty} \frac{\Gamma(s)}{(n+a)^s} = \Gamma(s)\zeta(s, a).$$

Thus, it suffices to show that the integral on the right defines a holomorphic function of s on $\sigma > 1$. To do this, we shall show analyticity in every strip $1 + \delta < \sigma < c$ where $\delta > 0$. Obviously the functions

$$F_N(s) = \int_0^N \frac{x^{s-1}e^{-ax}}{1 - e^{-x}}$$

are holomorphic on $\sigma > 1$. We shall show that they converge uniformly to the integral on the right hand side. Indeed, their difference is given by the integral

$$\left| \int_N^\infty \frac{x^{s-1}e^{-ax}}{1 - e^{-x}} \right| \leqslant \int_N^\infty \frac{x^{\sigma-1}e^{-ax}}{1 - e^{-x}}\, dx \leqslant \int_N^\infty x^{\sigma-1}e^{-ax}\, dx \leqslant \int_N^\infty x^{c-1}e^{-ax}\, dx.$$

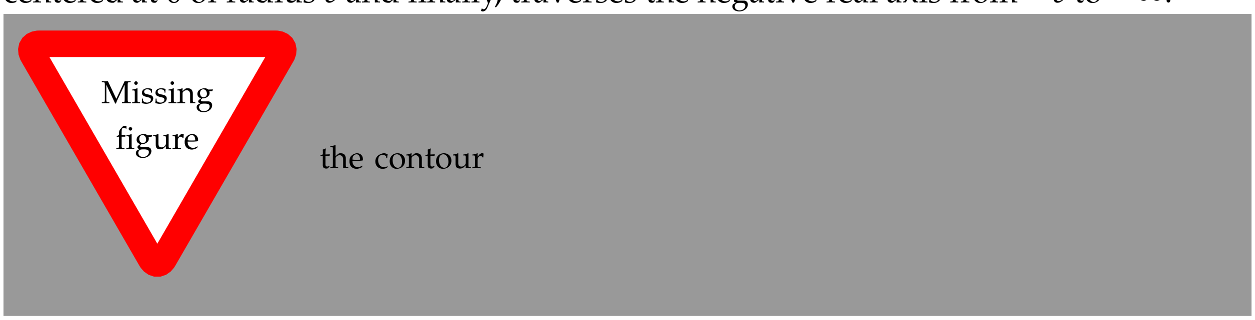
The uniform convergence thus follows from the fact that $\Gamma(c)$ is well defined and converges. ■

COROLLARY. In particular, for $a = 1$, we have

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}e^{-ax}}{1 - e^{-x}}$$

§§ Analytic Continuation of $\zeta(s, a)$

Let $0 < c < 2\pi$ and let C denote the piecewise smooth "contour" which first traverses the negative real axis from $-\infty$ to $-c$ and then traverses, in counter-clockwise sense, the circle centered at 0 of radius c and finally, traverses the negative real axis from $-c$ to $-\infty$.



Let C_1, C_2, C_3 denote the aforementioned smooth pieces of C. Then, C_1 is parametrized as $re^{-\pi i}$ for r running from ∞ to c. C_2 is parametrized in the obvious way and C_3 is parametrized as $re^{\pi i}$ for r running from c to ∞.

THEOREM 5.3. For $0 < a \leqslant 1$, the function defined by contour integral

$$I(s, a) = \frac{1}{2\pi i} \int_C \frac{z^{s-1}e^{az}}{1 - e^z}\, dz$$

is entire. Further, we have

$$\zeta(s, a) = \Gamma(1 - s) I(s, a) \qquad \text{for } \sigma > 1,\ \sigma \notin \mathbb{Z}.$$

Here, z^s means $r^s e^{-\pi i s}$ on C_1 and $r^s e^{\pi i s}$ on C_3.

Proof. Let $M > 0$ and consider the compact disk $|s| \leqslant M$. The integral can be broken up as $\int_{C_1} + \int_{C_2} + \int_{C_3}$. Since C_2 is a compact contour, the integral $\int_{C_2}$ defines an entire function anyway. Therefore, we need only show that the integrals corresponding to C_1 and C_3 are uniformly convergent on the chosen compact disk.

Along C_1, for $r \geqslant 1$, we have

$$|z^{s-1}| = r^{\sigma-1}\left|e^{-\pi i(\sigma-1+it)}\right| = r^{\sigma-1}e^{\pi t} \leqslant r^{M-1}e^{\pi M}$$

The same bound works on C_3. Therefore, on either C_1 or C_3, for $r \geqslant 1$, we have

$$\left|\frac{z^{s-1}e^{az}}{1-e^z}\right| \leqslant \frac{r^{M-1}e^{\pi M}e^{-ar}}{1-e^{-r}} = \frac{r^{M-1}e^{\pi M}e^{(1-a)r}}{e^r-1}.$$

For $r > \log 2$, we have $e^r - 1 > e^r/2$ and hence,

$$\left|\frac{z^{s-1}e^{az}}{1-e^z}\right| \leqslant 2r^{M-1}e^{\pi M}e^{-ar}.$$

Since the $\Gamma(M)$ exists, we conclude that the convergence of the integral along C_1 and C_3 is uniform. The argument is similar to the one in the previous proof.

Finally, we must show the identity. Let $g(z) = e^{az}/(1-e^z)$. We have

$$2\pi i I(s,a) = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) z^{s-1}g(z)\,dz.$$

That is,

$$\begin{aligned}2\pi i I(s,a) &= \int_\infty^c r^{s-1}e^{-\pi is}g(-r)\,dr + i\int_{-\pi}^{\pi} c^{s-1}e^{(s-1)i\theta}ce^{i\theta}g(ce^{i\theta})\,d\theta + \int_c^\infty r^{s-1}e^{\pi is}g(-r)\,dr \\ &= 2i\sin(\pi s)\int_c^\infty r^{s-1}g(-r)\,dr + ic^s\int_{-\pi}^{\pi} e^{is\theta}g(ce^{i\theta})\,d\theta.\end{aligned}$$

Set

$$I_1(s,c) = \int_c^\infty r^{s-1}g(-r)\,dr \quad \text{and} \quad I_2(s,c) = \frac{c^s}{2}\int_{-\pi}^{\pi} e^{is\theta}g(ce^{i\theta})\,d\theta.$$

Then,

$$\pi I(s,a) = \sin(\pi s)I_1(s,c) + I_2(s,c).$$

We claim that $\lim_{c\to 0} I_2(s,c) = 0$. Note that $g(z)$ is analytic in $|z| < 2\pi$ except for a simple pole at $z = 0$ and hence, $zg(z)$ is is analytic everywhere inside $|z| < 2\pi$. Consider the closed disk $|z| \leqslant \pi$. The function $zg(z)$ is analytic, hence, bounded on $|z| \leqslant \pi$, consequently, there is a constant $A > 0$ such that $|g(z)| \leqslant A/|z|$ for $|z| \leqslant \pi$. Then, for $c < \pi$, we have

$$|I_2(s,c)| \leqslant \frac{c^\sigma}{2}\int_{-\pi}^{\pi} e^{-t\theta}\frac{A}{c}\,d\theta \leqslant Ae^{\pi|t|}c^{\sigma-1},$$

and the conclusion follows.

Note that the integral remains unchanged upon changing the value of c, which follows from one of Cauchy's theorems. Now, note that

$$\lim_{c\to 0} I_1(s,c) = \int_0^\infty \frac{r^{s-1}e^{-ar}}{1-e^{-r}}\,dr = \Gamma(s)\zeta(s,a) \quad \text{for } \sigma > 1.$$

Hence, we have

$$\pi I(s,a) = \sin(\pi s)\Gamma(s)\zeta(s,a) \quad \text{for } \sigma > 1.$$

Recall Euler's reflection formula,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{for } s \in \mathbb{C}\backslash\mathbb{Z}.$$

Consequently, we have

$$\sin(\pi s)\Gamma(s) = \frac{\pi}{\Gamma(1-s)} \quad \text{for all } s \in \mathbb{C},$$

since $1/\Gamma(1-s)$ is an entire function. Substituting this, above, we have,

$$I(s,a) = \frac{1}{\Gamma(1-s)}\zeta(s,a).$$

When $\sigma \notin \mathbb{Z}$, we can rearrange the above in the required form. ■

Definition 5.4. For $\sigma \leqslant 1$, *define*

$$\zeta(s,a) = \Gamma(1-s)I(s,a).$$

Theorem 5.5. The function $\zeta(s,a)$ so defined is analytic for all s except for a simple pole at $s = 1$ with residue 1.

Proof. That it is analytic is obvious. We have

$$I(1,a) = \frac{1}{2\pi i}\int_C \frac{e^{az}}{1-e^z}\,dz.$$

In this case, the integrals on C_1 and C_3 cancel and we are left with

$$I(1,a) = \frac{1}{2\pi i}\int_{C_2} \frac{e^{az}}{1-e^z} = \text{Res}_{s=0}\frac{e^{as}}{1-e^s} = -1.$$

Consequently,

$$\text{Res}_{s=1}\,\zeta(s,a) = \lim_{s\to 1}(s-1)\Gamma(1-s)I(s,a) = -\,\text{Res}_{s=0}\,\Gamma(s)\times I(1,a) = 1. \tag{1}$$

This completes the proof. ■

§§ Hurwitz's Formula

Consider the Dirichlet series

$$F(x,s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}.$$

This converges absolutely in $\sigma > 1$ and hence, defines a holomorphic functions there. If $x \notin \mathbb{Z}$, then the series converges conditionally in $\sigma > 0$ and hence, is holomorphic there. In any case, $F(x,s)$ is periodic in x with period 1. We call this the *periodic Zeta function*.

LEMMA 5.6. For $0 < r < \pi$, let $S(r)$ denote the region that remains after removing all open circular disks of radius r centered at $2n\pi i$ for $n \in \mathbb{Z}$. If $0 < a \leqslant 1$, then the function

$$g(z) = \frac{e^{az}}{1 - e^z}$$

is bounded in $S(r)$. *The bound obviously depends on r.*

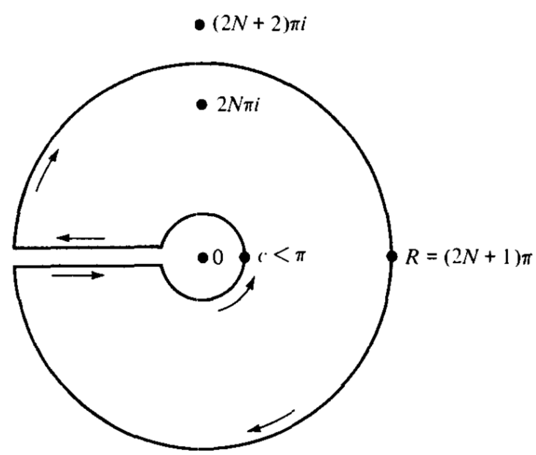
Proof. ∎ add

THEOREM 5.7 (HURWITZ'S FORMULA). If $0 < a \leqslant 1$ and $\sigma > 1$, then

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi i s/2} F(a,s) + e^{\pi i s/2} F(-a,s)\right).$$

If $a \neq 1$, this representation is valid in $\sigma > 0$.

Proof. For every positive integer N, let $C(N)$ denote the contour shown in the following figure.



Set

$$I_N(s,a) = \frac{1}{2\pi i}\int_{C(N)} \frac{z^{s-1}e^{az}}{1-e^z}\,dz$$

with the same conventions on z^{s-1} as mentioned while defining $I(s,a)$.

We first show that $\lim_{N\to\infty} I_N(s,a) = I(s,a)$ for $\sigma < 0$. To do this, it suffices to show that the integral along the outer circle vanishes as $N \to \infty$. Since the orientation of the outer cycle is irrelevant while showing this, we parametrize the outer circle as $z = Re^{i\theta}$ where $-\pi \leqslant \theta \leqslant \pi$. Consequently,

$$|z^{s-1}| = |R^{s-1}e^{i\theta(s-1)}| \leqslant R^{\sigma-1}e^{\pi|t|}.$$

Due to Lemma 5.6, there is an $A > 0$ (independent of N) such that the integrand is bounded by $AR^{\sigma-1}e^{\pi|t|}$. Thus, the integral can be bounded above in absolute vaule by

$$2\pi R^{\sigma}e^{\pi|t|}.$$

But since $\sigma < 0$, we have the desired conclusion as $N \to \infty$. We rewrite this as

$$\lim_{N\to\infty} I(1-s,a) = I(1-s,a) \quad \text{for } \sigma > 1.$$

We now use Cauchy's Residue Theorem to compute the value of $I_N(1-s,a)$. The poles corresponding to which the winding number is non-zero (in fact, precisely -1) are $2n\pi$ for $n \in \{-N,\dots,N\}\backslash\{0\}$.

Let

$$R(n) = \text{Res}_{z=2n\pi i}\,\frac{z^{-s}e^{az}}{1-e^z}.$$

Then,

$$R(n) = \lim_{z\to 2n\pi i}(z-2n\pi i)\frac{z^{-s}e^{az}}{1-e^z} = -\frac{e^{2n\pi ia}}{(2n\pi i)^s}.$$

Consequently, for $\sigma > 1$,

$$I_N(1-s,a) = \sum_{n=1}^{N}\frac{e^{2n\pi ia}}{(2n\pi i)^s} + \sum_{n=1}^{N}\frac{e^{-2n\pi ia}}{(-2n\pi i)^s} = \frac{e^{-\pi is/2}}{(2\pi)^s}\sum_{n=1}^{N}\frac{e^{2n\pi ia}}{n^s} + \frac{e^{\pi is/2}}{(2\pi)^s}\sum_{n=1}^{N}\frac{e^{-2n\pi ia}}{n^s}.$$

Taking $N \to \infty$, we get

$$I(1-s,a) = \frac{e^{-\pi is/2}}{(2\pi)^s}F(a,s) + \frac{e^{\pi is/2}}{(2\pi)^s}F(-a,s) \quad \text{for } \sigma > 1.$$

Recall that by definition, we have $\zeta(1-s,a) = \Gamma(s)I(1-s,a)$ for $\sigma > 0$, thus, for $\sigma > 1$. This gives,

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2}F(a,s) + e^{\pi is/2}F(-a,s)\right) \quad \text{for } \sigma > 1.$$

If $a \neq 1$, then the right hand side is analytic for $\sigma > 0$, as is the left hand side, whence the equality holds for $\sigma > 0$. This completes the proof. ∎

§§ Riemann's Functional Equation

THEOREM 5.8. For all $s \neq 0$, we have

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

Proof. Put $a = 1$ in Hurwitz's formula to get the identity, (for $\sigma > 1$)

$$\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi i s/2}F(1,s) + e^{\pi i s/2}F(1,s)\right) = \frac{\Gamma(s)}{(2\pi)^s}2\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

Let n be a positive integer and let $s \to 2n+1$. In this limit, the right hand side vanishes and hence, we have $\zeta(-2n) = 0$ for all positive integers n. Thus, the right hand side is a well defined function that is holomorphic (modulo removable singularities) on $\mathbb{C}\backslash\{0\}$. Further, since $\zeta(1-s)$ is holomorphic on $\mathbb{C}\backslash\{0\}$, equality holds for all $s \neq 0$. ■

From Gauß's multipliation formula, we get

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2\pi^{1/2}2^{-2s}\Gamma(2s)$$

whenever either of the two sides is defined. Put $s \mapsto (1-s)/2$ to get

$$2^s\pi^{1/2}\Gamma(1-s) = \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right),$$

whenever either of the two sides is defined.

The reflection formula gives

$$\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right) = \frac{2^{-s}\pi^{1/2}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$

whenever either of the two sides is defined.

We have

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

whenever either of the two sides is defined. Thus, we have

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Define the *xi function* as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

This is an entire function and satisfies the equation

$$\xi(s) = \xi(1-s).$$

This is known as *Riemann's functional equation*.

§§ Functional equation for L-functions

THEOREM 5.9. If h and N are positive integers with $1 \leqslant h \leqslant N$, then for all $s \neq 0$, we have

$$\zeta\left(1-s, \frac{h}{N}\right) = \frac{2\Gamma(s)}{(2\pi N)^s} \sum_{r=1}^{N} \cos\left(\frac{\pi s}{2} - \frac{2\pi r h}{N}\right) \zeta\left(s, \frac{r}{N}\right).$$

Proof. For $\sigma > 1$, note that

$$\begin{aligned} F\left(\frac{h}{N}, s\right) &= \sum_{n=1}^{\infty} \frac{e^{2\pi i n h/N}}{n^s} \\ &= \sum_{r=1}^{N} \sum_{q=0}^{\infty} \frac{e^{2\pi i r h/N}}{(qN+r)^s} \\ &= \frac{1}{N^s} \sum_{r=1}^{N} e^{2\pi i r h/N} \sum_{q=0}^{\infty} \frac{1}{\left(q+\frac{r}{N}\right)^s} \\ &= N^{-s} \sum_{r=1}^{N} e^{2\pi i r h/N} \zeta\left(s, \frac{r}{N}\right). \end{aligned}$$

Substituting this in Hurwitz's formula, we obtain the equality for $\sigma > 1$. The result holds for all $s \neq 0$ as a result of analytic continuation. ∎

Let χ be a Dirichlet character modulo N. Then, $L(s, \chi)$ is absolutely convergent for $\sigma > 1$. In this half plane, we can write

$$\begin{aligned} L(s, \chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\ &= \sum_{r=1}^{N} \sum_{q=0}^{\infty} \frac{\chi(r)}{(qN+r)^s} \\ &= \frac{1}{N^s} \sum_{r=1}^{N} \chi(r) \zeta\left(s, \frac{r}{N}\right). \end{aligned}$$

From the theory we developed earlier, we know that the Hurwitz zeta function has an analytic continuation to all of $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1.

- If χ is not the principal character modulo N, then $\sum_{r=1}^{N} \chi(r) = 0$ and hence, the right hand side of the above equation is entire. Consequently, $L(s, \chi)$ can be analytically continued to an *entire function*.
- On the other hand, if $\chi = \mathbb{1}$ is the principal character, then the right hand side has a simple pole at $s = 1$ of residue $\varphi(N)/N$.

PROPOSITION 5.10. Let χ be a primitive character modulo N. Then,

$$G(1, \overline{\chi}) L(s, \chi) = \sum_{h=1}^{N} \overline{\chi}(h) F\left(\frac{h}{N}, s\right) \qquad \text{for } \sigma > 1.$$

Proof. Omitted owing to its obviousness. The primitive-ness of the character is required only to use the fact that the Gauss sum is separable. ■

THEOREM 5.11 (FUNCTIONAL EQUATION FOR L-SERIES). Let χ be a primitive character modulo N. Then, for all s, we have

$$L(1-s,\chi) = \frac{N^{s-1}\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\right) G(1,\chi)L(s,\overline{\chi}).$$

Proof. Hurwitz's formula says

$$\zeta(1-s,h/N) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2}F(h/N,s) + e^{\pi is/2}F(-h/N,s)\right) \qquad \text{for } \sigma > 1.$$

Thus, for $\sigma > 1$,

$$\sum_{h=1}^{N}\chi(h)\zeta\left(1-s,\frac{h}{N}\right) = \frac{\Gamma(s)}{(2\pi)^s}\left\{e^{-\pi is/2}\sum_{h=1}^{N}\chi(h)F(h/N,s) + e^{\pi is/2}\sum_{h=1}^{N}\chi(h)F(-h/N,s)\right\}.$$

We simplify the second term,

$$\sum_{h=1}^{N}\chi(h)F(-h/N,s) = \sum_{h \bmod N}\chi(h)F\left(\frac{N-h}{N},s\right) = \chi(-1)\sum_{h \bmod N}\chi(h)F(h/N,s).$$

Substituting this back, for $\sigma > 1$, the right hand side becomes

$$\frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\right)\sum_{h \bmod N}\chi(h)F(h/N,s).$$

Using Proposition 5.10, the above simplifies as

$$\sum_{h=1}^{N}\chi(h)\zeta\left(1-s,\frac{h}{N}\right) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\right) G(1,\chi)L(s,\overline{\chi})$$

for $\sigma > 1$. The left hand side is holomorphic on $s \neq 0$, as is the right hand side (since $\overline{\chi}$ is non principal). Thus, the equality holds for all s (since the right hand side is entire). In particular, we can suppose $\Re(s) < 0$, whence, we can multiply by N^{s-1} to obtain the equality

$$L(1-s,\chi) = \frac{N^{s-1}\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\right) G(1,\chi)L(s,\overline{\chi}).$$

This equality holds in $\sigma < 0$ and hence, everywhere, since both sides are entire. This completes the proof. ■

§6 The Prime Number Theorem

Lemma 6.1. $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R} \setminus \{0\}$.

Lemma 6.2. The series

$$\Phi(s) = \sum_{n \geqslant 2} \sum_{p} \frac{1}{np^{ns}}$$

converges uniformly on compacta to a holomorphic function on $\Re s > \frac{1}{2}$.

Proof. Let $s = x + iy$ with $x > \frac{1}{2}$. We have the inequalities

$$\sum_{n \geqslant 2} \sum_{p} \frac{1}{np^{nx}} = \sum_{p} \frac{1}{p^{2x}} \left(\sum_{n \geqslant 0} \frac{1}{(n+2)p^{nx}} \right) \leqslant \sum_{p} \frac{1}{p^{2x}} \left(\sum_{n \geqslant 0} \frac{1}{\sqrt{2}^{n}} \right)$$

and the conclusion follows. ■

Define the series

$$L(s) = \sum_{p} \frac{1}{p^s},$$

which is easily seen to be holomorphic in $\Re s > 1$ as the series converges uniformly on compacta. Let

$$\ell(s) = \sum_{p} \frac{\log p}{p^s} = -L'(s)$$

on Res > 1.

Notice that

$$L(s) = \log \zeta(s) - \Phi(s) \quad \text{for } \Re s > 1.$$

Due to Lemma 6.1, the function $(s-1)\zeta(s)$, which is known to be entire, does not vanish on an open set containing $\{z \colon \Re z \geqslant 1\}$. Therefore, we may consider a logarithm for the same around $s = 1$. It follows that on the right half plane $\Re s > 1$,

$$\ell(s) - \frac{1}{s-1} = -\left(L(s) + \log(s-1)\right)' = -\left(\log\left((s-1)\zeta(s)\right) - \Phi(s)\right)'.$$

Note that the right hand side is defined and analytic in a neighborhood of $s = 1$ and hence, $\ell(s) - \frac{1}{s-1}$ is defined and analytic in an open set containing $\Re s \geqslant 1$. This will be very useful later on.

Lemma 6.3. Let $f : [0, \infty) \to \mathbb{C}$ be a bounded, locally integrable function. Define $g : \{z \colon \Re z > 0\} \to \mathbb{C}$ by

$$g(z) = \int_0^\infty e^{-zt} f(t)\, dt.$$

Then g is well-defined and analytic on its domain of definition.

Proof. Define $g_T : \mathbb{C} \to \mathbb{C}$ by

$$g_T(z) = \int_0^T e^{-zt} f(t)\, dt.$$

We shall show that $g_T \to g$ uniformly on compacta contained in the right half plane. Indeed, let K be one such compact set. Then, there is a $\delta_0 > 0$ such that $\Re z \geqslant \delta_0$ for every $z \in K$. It follows that for $T < S$,

$$|g_S(z) - g_T(z)| \leqslant \int_T^S e^{-\delta_0 t} |f(t)|\, dt,$$

which goes to zero since f is bounded. Thus, g is analytic on its domain of definition. ■

THEOREM 6.4 (NEWMAN). Let $f : [0, \infty) \to \mathbb{C}$ be a bounded, locally integrable functionand suppose that

$$g(z) = \int_0^\infty e^{-zt} f(t)\, dt \quad \Re z > 0,$$

extends analytically to an open set containing $\Re z \geqslant 0$. Then, $\int_0^\infty f(t)\, dt$ exists and is equal to $g(0)$.

LEMMA 6.5. Suppose $h : [1, \infty) \to \mathbb{R}$ is a non-decreasing function and

$$\int_1^\infty \frac{h(x) - x}{x^2}\, dx$$

converges. Then, $h(x) \sim x$.

Proof. Suppose for some $\lambda > 1$, there are arbitrarily large values of x with $h(x) \geqslant \lambda x$. Then,

$$\int_x^{\lambda x} \frac{h(t) - t}{t^2}\, dt \geqslant \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\, dt = \int_1^\lambda \frac{\lambda - s}{s^2}\, ds > 0$$

for all such x (which are arbitrarily large), a contradiction to the fact that the integral converges.

Similarly, if for some $\lambda < 1$, there are arbitrarily large values of x with $h(x) \leqslant \lambda x$, then

$$\int_{\lambda x}^{x} \frac{h(t) - t}{t^2} \leqslant \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2}\, dt = \int_\lambda^1 \frac{\lambda - s}{s^2}\, dt < 0$$

for all such x (which are arbitrarily large). This is again a contradiction. ■

It is not hard to see the equality

$$\ell(s) = s \int_0^\infty e^{-st} \vartheta(e^t)\, dt \qquad \Re s > 1,$$

which follows by just integrating the function $t \mapsto \vartheta(e^t)$ step-wise. Thus,

$$\frac{\ell(s+1)}{s+1} - \frac{1}{s} = \int_0^\infty e^{-st} \left(e^{-t} \vartheta(e^t) - 1 \right)\, dt \qquad \Re s > 0.$$

Set $g(s) = \frac{\ell(s+1)}{s+1} - \frac{1}{s}$ and $f(t) = e^{-t}\vartheta(e^t) - 1$. Then, f is a bounded locally integrable function on $[0, \infty)$ and g is holomorphic in a neighborhood of $\Re s \geqslant 0$. Due to Theorem 6.4, it follows that

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2}\, dx$$

converges. Finally, using Lemma 6.5, we have $\vartheta(x) \sim x$, which is equivalent to the Prime Number Theorem due to Theorem 2.5.